Large Cardinals and Forcing: the Levy-Solovay Theorem

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1 Overview

At the International Congress of Mathematicians in 1900, David Hilbert presented a list of problems that would guide mathematical research for the next century. Sitting at number one is the *Continuum Hypothesis*(CH), which is the statement that every non-empty set of real numbers is either in bijection with the natural numbers, or in bijection with the whole set of real numbers. A surprising discovery about half a century later was that the CH was neither provable nor disprovable from the axioms of ZFC (and *this fact* can be proven in ZFC).

At the same time, Kurt Gödel proposed the following program of research, now known as "Gödel's Program":

> Gödel's Program: to decide (i.e., prove or disprove) mathematically interesting problems by adopting well-justified axioms beyond the axioms of ZFC.

Chief among these "well-justified new axioms" are the stronger axioms of infinity: the axioms stating the existence of what's called *large cardinals*. These are mathematical objects that are so large and complex, that ZFC itself alone cannot prove that they exist.

A natural attempt, then, is to find a large cardinal axiom A such that in $\mathsf{ZFC} + A$, we can either prove or disprove CH. While this might sound promising, it was rendered utterly hopeless by the landmark result of Levy and Solovay in the 1960s, which roughly says that no reasonable large cardinal axiom can decide CH.

In this course, we will learn the proof of the Levy-Solovay theorem. To do this, we need knowledge in four key areas: 1. basic set theory, 2. models of set theory, 3. theory of large cardinals, and 4. the method of forcing. The following is an outline of the course:

1. Axioms of ZFC, ordinals and cardinals, transfinite recursion and induction; review of formal logic: completeness and compactness, Löwenheim-Skolem theorems (1 week)

2. cardinal arithmetic: addition, multiplication, exponentiation. Cofinality: regular and singular cardinals (1 week)

3. Models of ZFC, satisfaction relation, absoluteness, the reflection principle, Łoś's theorem, Tarski's undefinability theorem, Levy hierarchy of formula complexity (2 weeks)

4. "small" large cardinals: inaccessible and Mahlo cardinals, and the consistency hierarchy generated by them (1 week)

5. "large" large cardinals: measurable cardinal, its definition in terms of normal measures and in terms of elementary embeddings (2 weeks)

6. Forcing: the metamathematical set-up, filters and the Rasiowa-Sikorski lemma (1 week)

7. More forcing posets, forcing construction: $\mathbb{P}$-names and interpretation by a generic filter (2 weeks)
8. basic properties of forcing extensions, definability of the forcing relation (definability lemma and truth lemma) (2 weeks)
9. poset combinatorics: chain condition, closure condition, and the Delta-system lemma, preservation of cardinals (1 week)
10. forcing $\neg\mathsf{CH}$ and forcing CH (1 week)
11. interaction between forcing and large cardinals, the original Levy-Solovay theorem about measurable cardinals (1 week)

2 Format of the course

This course will be modeled after a typical REU program. There will be 15 weekly meetings, each lasting 1.5-2 hours. The content covered in the course is usually worth two semesters of graduate coursework in set theory, so to compress everything down to 15 weekly meetings, heavy student investment is needed. This means that the student will (try their best to) work through the assigned reading each week and the meeting sessions will be the instructor lecturing on important lemmas/theorems that deserve more attention and Q&A. The student is welcome to ask questions outside of lecture (i.e., other times during the week) as well.

3 Expected output

At the end of the course, the student is expected to produce a REU-style expository paper. Examples of such papers will be provided. At the bare minimum, the paper should cover the proof of the Levy-Solovay theorem.

4 Some resources

Two set theory textbook that every set theorist has are [Kun80] and [Jec13]. There's a newer version of Kunen as well [Kun14]; the old version and the new version are equally good. The standard reference on large cardinals is Kanamori's [Kan08]. We will mostly follow [Jec13], except the forcing part, where we will use Kunen's exposition. Lecture notes composed by the instructor will be distributed throughout the course.

There are two books devoted to introducing forcing to beginners: [Wea14] and [Dža20]. Weaver's book contains a lot of examples of applications of forcing to other areas of mathematics; however it is not recommended because it uses terminology and notation that deviates from most set theory literature, which can make things confusing.

References

[Dža20] Mirna Džamonja. *Fast Track to Forcing*. Vol. 98. Cambridge University Press, 2020.

[Jec13] Thomas Jech. *Set theory*. Springer Science & Business Media, 2013.

[Kan08] Akihiro Kanamori. *The higher infinite: large cardinals in set theory from their beginnings*. Springer Science & Business Media, 2008.

[Kun80] Kenneth Kunen. *Set Theory: An Introduction to Independence Proofs*. North-Holland, 1980.

[Kun14] Kenneth Kunen. *Set theory an introduction to independence proofs*. Elsevier, 2014.

[Wea14] Nik Weaver. *Forcing for mathematicians*. World Scientific, 2014.