Prerequisite Material for Fudan 2022 Summer School

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This note contains reminders for roughly the following:

1. transitive models
2. measurable cardinals
3. reflection
4. constructibility
5. ordinal definability

The intended audience is participants of the Fudan 2022 Logic Summer School, course on large cardinals beyond choice, taught by Gabriel Goldberg.

This is more of a collection than presentation. I assume most readers have seen at least parts of the following at least once. If not, the material is standard and can be found in standard textbooks on set theory. Off we go...

1 Transitive Models

Definition 1. A transitive model of some theory T is a transitive set or proper class M, such that $(M, \in) \vDash T$.

Remark 2. Transitive models are usually gotten from the following ways:

1. by assumption ("assume T has a transitive model...")
2. follows from large cardinals (typically in the forms of $V_\kappa, H_\kappa, L_\kappa$ or elementary submodels of them)
3. inner models construction ($L, \mathrm{HOD}, L(\mathbb{R}), L[X], \mathrm{HOD}(\mathbb{R})$, and so on)
4. forcing construction (makes another trnsitive model out of one that you already have)

Another way of getting transitive models is by collapsing a well-founded model. Recall that a structure (M, E) is well-founded (with respect to the relation E), iff there is no infinite descending E-chain. To collapse a well-founded model is to apply the following tool:

Theorem 3. (Mostowski Collapse) Let (X, E) be a (possibly proper class) structure, where $E \subseteq X \times X$ satisfies the following conditions:
1. (X, E) is extensional, i.e., $\forall a, b \in X \ a = b \iff (\forall c \in X \ cEa \iff cEb)$
2. E is well-founded
3. E is set-like, i.e., for each $a \in X$, the class $\text{ext}(a) = \{b \in X \mid bEa\}$ is a set.
Then there is a unique transitive M and a unique isomorphism $\pi : (X, E) \to (M, \in)$

Transitive models of $\mathsf{ZF(C)}$ look a lot like V. The following facts will make this claim more precise.

Definition 4. A formula φ is said to be absolute for a transitive model M iff,

$$\forall x_1, ..., x_n \in M \ \varphi^M(x_1, ..., x_n) \leftrightarrow \varphi(x_1, ..., x_n)$$

where φ^M denotes φ with quantifiers restricted to M. φ is said to be upward-absolute iff $\forall x_1, ..., x_n \in M \ \varphi^M(x_1, ..., x_n) \to \varphi(x_1, ..., x_n)$; downward absolute iff $\forall x_1, ..., x_n \in M \ \varphi(x_1, ..., x_n) \to \varphi^M(x_1, ..., x_n)$

Fact 5.
1. Δ_0 formulas are absolute for all transitive classes (or absolute for short)
2. Σ_1 formulas are upward-absolute.
3. Π_1 formulas are downward-absolute.
4. Δ_1 formulas are absolute.

Remark 6. The above facts relate truth in M and truth in V. Sometimes it is necessary to consider the relation between truths of two models $M \subseteq N$. Definition of absoluteness is modified in the obvious way.

A general fact about transitive models of ZFC is that they are entirely determined by what sets of ordinals they have (recall that under ZFC, every set is coded by a set of ordinals).

Proposition 7. Let M, N be transitive models of ZF with the same ordinals, such that for every α, $\mathcal{P}^M(\alpha) = \mathcal{P}^N(\alpha)$. Let at least one of them satisfy Choice. Then $M = N$

Proof. Without loss of generality, suppose M satisfies AC. Since Göddel's pairing function $\Gamma : ON \times ON \to ON$ is absolute, we note that for every ordinal α, we have $\mathcal{P}(\alpha \times \alpha)^M = \mathcal{P}(\alpha \times \alpha)^N$ as well.

First we show $M \subseteq N$. Let $X \in M$. By $(AC)^M$, we see that there is some $f \in M$ such that $f : \theta \leftrightarrow \text{tr cl}(\{X\})$ for some ordinal $\theta \in M$. Now we define the relation E on θ as follows: $\alpha E \beta \iff f(\alpha) \in f(\beta)$. Now E is just a set of ordered pairs of ordinals in M, so by assumption $E \in N$. In N, we may take the Mostowski collapse of (θ, E) and get an isomorphic $(T, \in)$. But notice that this is just $\text{tr cl}(\{X\})$. Now since $X \in \text{tr cl}(\{X\}) \in N$ and N is transitive, we have $X \in N$.

Now we show $N \subseteq M$. We will prove this by $\in$-induction. Let $X \in N$ and suppose that the claim holds for every $x \in X$ (i.e., $X \subseteq M$). Take $Y \in M$ such that $X \subseteq Y$. So we consider $f \in M$ which well-orders Y. Since $M \subseteq N$, this f will be in N and hence $f(X) \in N$. But $f(X) \subseteq ON$, so $f(X) \in M$. Taking $f^{-1}[X]$, we have that $X \in M$. $\square$

2 Measurable Cardinals

Definition 8. Let X be a set, a *filter* on X is a collection U of subsets of X, satisfying

1. $X \in U$
2. $\emptyset \notin U$
3. $(A \in U \wedge B \in U) \rightarrow A \cap B \in U$
4. $(A \in U \wedge B \supseteq X) \rightarrow B \in U$

Moreover, U is an *ultrafilter* iff for every $Y \subseteq X$, either $Y \in U$ or $(X \setminus Y) \in U$.

Remark 9. First, in the present context, this definition is usually applied to ordinals/cardinals or some related objects.

Second, Observe that if U is a filter on a cardinal κ, then $U \subseteq \mathcal{P}(\kappa)$; so: $U \in \mathcal{P}\mathcal{P}(\kappa)$. For a cardinal κ, if there is a filter U on κ, then $U \in V_{\kappa+2}$.

Definition 10. Let κ be a cardinal and let U be a filter on κ. We say U is *nonprincipal*, if U doesn't contain any singleton. That is, U is *nonprincipal* iff there is no $\{\alpha\} \subseteq \kappa$ such that $\{\alpha\} \in U$.

We say U is κ-complete, if the following holds:
For any $\lambda < \kappa$, if we have $\{X_\alpha \mid \alpha < \lambda\}$ all in U (this means: $X_0 \in U, X_1 \in U, X_2 \in U, ...$), then $\bigcap_{\alpha<\lambda} X_\alpha \in U$.

Definition 11. (Ultrapower)
Given a structure M, and an ultrafilter U on κ, we define the following relations on the product $\Pi_{i\in\kappa} M := \{f \mid f : \kappa \to M\}$:

$$\begin{aligned} f =^* g &\iff \{i \in S \mid f(i) = g(i)\} \in U \\ f \in^* g &\iff \{i \in S \mid f(i) \in g(i)\} \in U \end{aligned}$$

We define equivalence classes of $\Pi_{i\in\kappa} M$ modulo $=^*$:

$$[f] = \{g \mid f =^* g \wedge \underline{\forall h(h =^* f \to \operatorname{rank}(g) \leq \operatorname{rank}(h))}\}$$

(Note: the underlined clause is called "Scott's trick". This ensures the $[f]$'s are sets. Exercise: show that, given an $f : \kappa \to V$, the class $\{g \mid f =^* g\}$ is not a set. Hint: you can always modify f on 1 single input, without altering whether $f =^* g$.)
The *ultrapower* of M is the structure of these equivalence classes $\mathrm{Ult}_U(M) = \Pi_{i\in\kappa} M/U = \{[f] : f \in \prod_{i\in\kappa} M\}$. Presently we are concerned with the case where M is the von Neumann universe V. Sometimes Ult is used as the shorthand for $\mathrm{Ult}_U(V)$, or its Mostowski collapse when it's well-founded.

Fact 12.
1. if U is a κ-complete nonprincipal ultrafilter on κ, then Ult is set-like and well-founded. This implies that it is isomorphic to a transitive model $M \subseteq V$. When no ambiguity arises, we write M and Ult and $\mathrm{Ult}_U(V)$ interchangeably.

2. Ult is elementarily equivalent to V (they satisfy the same sentences). This follows from Łoś's Theorem.

Definition 13. An elementary embedding is a function $j : M \to N$ such that for all $\vec{x} \in M$ and for all formulas $\varphi(\vec{x})$: $M \models \varphi(\vec{x}) \leftrightarrow N \models \varphi(j(\vec{x}))$

Claim 14. The function $j_U : V \to \text{Ult}$ befined by $j_U(a) = [c_a]$ is an elementary embedding, where for each $a \in V$, $[c_a]$ is the equivalence class of the constant functions $c_a(i) = a$ which map every index i to a.

Proof. Let $a \in V$, then by Łoś' theorem, $V \models \varphi(a)$ iff $\{i \in \kappa \mid V \models \varphi(c_a(i))\} \in U$ iff Ult $\models \varphi(c_a)$ iff Ult $\models \varphi([j(a)])$. $\square$

Definition 15. The critical point of j, written as $\text{crit}(j)$, is the least ordinal moved by j.

From an ultrafilter, we get an elementary embedding. Conversely, from an elementary embedding, we get an ultrafilter too.

Definition 16. Let U be an nonprincipal ultrafilter on κ. We say U is *normal* if it satisfies any one of the following equivalent definitions:

1. U is closed under diagonal intersection.
2. The U-almost everywhere version of Fodor's theorem holds: if $f : \kappa \to \kappa$ is such that $\{i < \kappa \mid f(i) < i\} \in U$, then there is some $\alpha < \kappa$ such that $\{i < \kappa \mid f(i) = \alpha\} \in U$.
3. the equivalence class of the identity function $id : \kappa \to \kappa$ gets represented as κ in the ultrapower. That is, $[\text{id}]_U = \kappa$.

Theorem 17. If $j : V \to M$ is a nontrivial elementary embedding and $\text{crit}(j) = \kappa$, then the set $U \subseteq \mathcal{P}(\kappa)$ defined by

$$X \in U \Leftrightarrow \kappa \in j(X)$$

is a normal κ-complete nonprincipal ultrafilter on κ. The U defined in this way is sometimes called *the normal measure derived from* j, or *the derived measure*.

Theorem 18 ((Facts about ultrapower embedding))**.** Let κ be a measurable cardinal, witnessed by ultrafilter U. Let $j : V \to M$ be the ultrapower embedding. Then:

1. j is the identity when restricted to V_κ.
2. $\text{crit}(j) = \kappa$ and $j(\kappa) > \kappa$.
3. κ must be inaccessible.
4. $V_{\kappa+1} = (V_{\kappa+1})^M$, and $\kappa^+ = (\kappa^+)^M$.
5. $2^\kappa \leq (2^\kappa)^M < j(\kappa) < (2^\kappa)^+$
6. ((dis-)continuity points of j) if λ is a limit ordinal with $\text{cof}(\lambda) \neq \kappa$, then $j(\lambda) = \sup_{\alpha<\lambda} j(\alpha)$; if $\text{cof}(\lambda) = \kappa$, then $j(\lambda) > \sup_{\alpha<\lambda} j(\alpha)$

7. (fixed points of j) if λ is a strong limit cardinal with $\mathrm{cof}(\lambda) \neq \kappa$, then $j(\lambda) = \lambda$.

8. ${}^{\kappa}M \subseteq M$, but ${}^{\kappa^+}M \not\subseteq M$.

9. $U \notin M$.

Proof. See, for example, Lemma 17.9 in Jech's Set Theory. $\square$

Corollary 19. If $j : V \to M$ is the ultrapower embedding from some ultrafilter U on κ, then $M \neq V$.

3 Reflection

With a measurable cardinal and an elementary embedding, we can carry out what's called reflection arguments. Here's an example:

Theorem 20. If κ is measurable, then there must be some inaccessible cardinal $\lambda < \kappa$.

Proof. Let $j : V \to M$ be an elementary embedding with $\mathrm{crit}(j) = \kappa$. We know $V \models$ "κ is inaccessible", but also inaccessibility is Π_1, hence $M \models$ "κ is inaccessible". This implies $M \models \exists \lambda < j(\kappa)$"$\lambda$ is inaccessible". Pulling this fact back to V by elementarity of j, we get $V \models \exists \lambda < \kappa$"$\lambda$ is inaccessible". $\square$

It is not hard to generalize this to show that there are $2, 3, 4, \ldots$ many inaccessible cardinals below κ.

This fact can be generalized:

Theorem 21. If κ is measurable, and $j : V \to M$ with $\mathrm{crit}(j) = \kappa$. If U the normal measure derived from j, then $\{\lambda < \kappa \mid \lambda \text{ is inaccessible }\} \in U$.

Proof. Whether κ is inaccessible is decided in $V_{\kappa+1}$ (why?). But we have that $V_{\kappa+1} = (V_{\kappa+1})^M$. So $M \models \kappa$ is inaccessible. Let $I := \{\lambda < \kappa \mid \lambda \text{ is inaccessible }\}$. Now, κ is an inaccessible below $j(\kappa)$, that is $\kappa \in j(I)$. By definition of U, we have $I \in U$. $\square$

The above is a typical example of a *reflection argument.* In a reflection argument, we use the structural relation between M and V to extract information about *kappa* or V_κ. Here's another example:

Theorem 22. Let κ be measurable with normal measure U. If $\{\lambda < \kappa \mid 2^\lambda = \lambda^+\} \in U$, then $2^\kappa = \kappa^+$.

Proof. Let $j_U : V \to M$ be the ultrapower embedding by U. Begin by recalling that, by normality, κ is represented by $[\mathrm{id}]_U$ in M. So,

$$M \models 2^\kappa = \kappa^+ \Leftrightarrow M \models 2^{[\mathrm{id}]} = [\mathrm{id}]^+ \Leftrightarrow \{i < \kappa \mid 2^{\mathrm{id}(i)} = \mathrm{id}(i)^+\} \in U \Leftrightarrow \{\lambda < \kappa \mid 2^\lambda = \lambda^+\} \in U$$

Finally, since M and V agree on what 2^κ and κ^+ are, we have that (actually in V) $2^\kappa = \kappa^+$. $\square$

4 Constructibility

Definition 23. Let $\mathcal{D}$ denote the "first-order definable powerset" operation. Given a set M, a first-order definable subset $X \subseteq M$ is a set of the form $\{a \in M \mid (M, \in) \vDash \varphi(a, \vec{b})\}$, where φ is a formula in the first-order language of set theory and parameters $\vec{b}$ are in M. For any set M, $\mathcal{D}(M) = \{X \subseteq M \mid X$ is a first-order definable subset of $M\}$. Then, $L = \bigcup_{\alpha \in \mathrm{Ord}} L_\alpha$, where

$$
\begin{aligned}
&L_0 = \emptyset \\
&L_{\alpha+1} = \mathcal{D}(L_\alpha) \\
&L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha, \text{ for limit } \lambda
\end{aligned}
$$

Definition 24. The constructible hierarchy $L[x]$ relative to a set x is defined in the same way as L, except at successor stages we consider $D_x(M) = \{X \subseteq M \mid X$ is definable over $(M, \in, x \cap M)\}$.

Fact 25.
1. $L \vDash \mathsf{ZF} + \mathsf{AC} + \mathsf{GCH}$.
2. $L[x]$ has a global well-ordering that is definable from x.
3. L is the minimal inner model. If M is any inner model, then $L \subseteq M$.
4. In general, $L[x]$ is the minimal inner model that can "see" x: if M is an inner model and $x \cap M \in M$, then $L[x] \subseteq M$

Definition 26. Let x be a set.

$$
\begin{aligned}
&L_0(x) = \mathrm{trcl}(\{x\}) \\
&L_{\alpha+1}(x) = \mathcal{D}(L_\alpha)(x) \\
&L_\lambda(x) = \bigcup_{\alpha<\lambda} L_\alpha, \text{ for limit } \lambda
\end{aligned}
$$

Fact 27.
1. $L(x) \vDash \mathsf{ZF}$
2. $L(x)$ is the minimal inner model that contains x as member.

We can only be sure that $L(x) \vDash \mathsf{ZF}$. The most common example of this kind of constructible universe is $L(\mathbb{R})$. For example, $L(\mathbb{R})$ does not satisfy the axiom of choice after adding ω_1 Cohen reals.

The famous theorem of Scott shows that measurable cardinals refute $V = L$.

Theorem 28 (Scott)**.** If there is a measurable cardinal, then $V \neq L$.

Proof. Let κ be the least measurable cardinal, and let U be an ultrafilter on it. Let $j : V \to M$ be the ultrapower embedding from U. If $V = L$, then this is really an elementary embedding $j : V \to V$. But the key point here is that expressions involving "j" can be expressed in the language of set theory (just say "this and that sets are in U").

So $V \vDash$ "κ is the least measurable cardinal", now $M \vDash$ "$j(\kappa)$ is the least measurable cardinal". But the assumption $V = L$ implies $M = V$. So this contradicts the fact that $j(\kappa) > \kappa$. □

5 Ordinal Definability

Definition 29. A set X is ordinal-definable iff there is a formula $\varphi(x, \vec{y})$ such that there are ordinals $\vec{\alpha}$, satisfying

$$X = \{x \mid \varphi(x, \vec{\alpha})\}$$

Remark 30. Formally, we will need to use the reflection theorem and say something is ordinal-definable if there is some V_β such that it is ordinal-definable (in the sense above) in V_β. But in practice it is easier to think of it in terms of the first "definition".

Definition 31. A set X is hereditarily ordinal-definable, iff $\mathrm{trcl}(\{X\})$ is ordinal definable. We write HOD for the class of hereditarily ordinal definable sets.

Definition 32. HOD_z is the class of sets hereditarily ordinal-definable from elements of z. That is, $X \in \mathrm{OD}_z$ iff there is some formula $\varphi(x, \vec{y}, \vec{v})$ such that there are ordinals $\vec{\alpha}$ and elements $\vec{z}$ from z, satisfying

$$X = \{x \mid \varphi(x, \vec{\alpha}, \vec{z})\}$$

And $X \in \mathrm{HOD}_z$ iff $\mathrm{trcl}(\{X\}) \in \mathrm{OD}_z$.

Fact 33. In general, if z is ordinal definable from z, then $\mathrm{HOD}_z \vDash \mathsf{ZF}$.

Fact 34. There is a (parameter-free) definable well-ordering of the universe in HOD. In fact, if M is an inner model with a definable well-ordering of the universe, then $M \subseteq \mathrm{HOD}$. So in some informal sense, HOD is the "largest" inner model that is nice.

Fact 35. We can define a relation $D(x, y) = z$ in the language of set theory, such that $D(\alpha, a) = b$ holds iff α codes a finite sequence of ordinals, the first one is a natural number coding a formula, which, along with the rest of that sequence and x as parameters, defines y.

One way of making things ordinal-definable is by *coding into the continuum pattern.*

Definition 36. Recall that if x is a set, then there is some ordinal δ_x and relation $E_x \subseteq \delta \times \delta$ such that $(tc(\{x\}), \in) \cong (\delta_x, E_x)$ Let α, λ be ordinals. Let $g : ON \to ON \times ON$ be the inverse of Gödel's pairing function. Define the set $c(\alpha, \lambda) \subseteq \lambda$ as follows: for all $i \in \lambda$, $i \in c(\alpha, \lambda) \iff 2^{\aleph_{\alpha+i+1}} = \aleph_{\alpha+i+2}$. We say that x *is coded into the continuum pattern at* α *with length* λ iff $g''c(\alpha, \lambda) = E_x$. That is, $\in \restriction tc(\{x\}) \cong g''c(\alpha, \lambda)$.

We say that x *is coded into the continuum pattern* when there is some α, λ such that x is coded into the continuum pattern at α with length λ.

Theorem 37. There is a forcing extension over L, so that in the extension, $L \neq \mathrm{HOD}$.

very sketchy proof. First, add a Cohen real to L and get $L[c]$. Then, use the homogeneity of Cohen forcing to argue that $c \notin \mathrm{HOD}$. So in $L[c]$, $L = \mathrm{HOD}$.

But now, for $i \in \omega$, use Easton forcing to make the continuum hypothesis true or false at $\aleph_i$ according to whether $i \in c$. That is, get a forcing extension $L[c][G]$ where $2^{\aleph_i} = \aleph_{i+1}$ iff $i \in c$.

So now, in $L[c][G]$, the Cohen real c is ordinal-definable by defining it to be the unique real number whose characteristic function behaves the same way as the truth of the generalized continuum hypothesis below $\aleph_\omega$. $\square$

Sometimes HOD_z can have nice structure coming from V:

Theorem 38 (Solovay, assumes AC in V)**.** $\mathrm{HOD}_{\mathbb{R}} \models \mathsf{DC}$.

Let A be a set. Let $R \subseteq \mathrm{HOD}_{\mathbb{R}}$ be a relation on some set A satisfying the assumption of DC.

Proof. We inductively define the following function $f : \omega \to \mathrm{HOD}_{\mathbb{R}}$ in V. Let $f(0)$ be some element of A.

Let α_0 be the least ordinal such that there is a real number x with $D(\alpha_0, x) = f(0)$, and let x_0 be such that $D(\alpha_0, x_0) = f(0)$.

Let α_{n+1} be the least ordinal α such that there is some real number x with $(f(n), D(\alpha, x)) \in R$. Let $x_{n+1} \in \mathbb{R}$ witness that, and let $f(n+1) := D(\alpha_{n+1}, x_{n+1})$.

We used AC to get the sequence of real numbers $(x_n \mid n \in \omega)$, and we code it as a single real r. Then The dependent choice function f is $\mathrm{HOD}_{\mathbb{R}}$ using r as a parameter, basically with the same definition above in the proof.

□

Finally, here's a less useful definition of HOD, but this makes it more similar to L.

Theorem 39 ((AC))**.** HOD can be obtained by the following recursion, where D_2 is second-order definable powerset:

$$\begin{aligned} N_0 &= \emptyset \\ N_{\alpha+1} &= \mathcal{D}_2(L_\alpha) \\ N_\lambda &= \bigcup_{\alpha<\lambda} L_\alpha, \text{ for limit } \lambda \end{aligned}$$

A set is in HOD iff it is in one of the N_α.