Measurable Cardinals and Ultrapower Embeddings

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In this note, we look at the connections between measurable cardinals, ultrapowers, elementary embeddings, and the constructible universe L.

0 Preliminary

We take some background facts and notions for granted. Here is a list of them.

Definition 0.1. (Gödel's Constructible Unvierse L) L is defined recursively, where $\mathcal{D}$ denotes the definable power set operator:

$$L_0 = \emptyset$$
$$L_{\alpha+1} = \mathcal{D}(L_\alpha)$$
$$L_\lambda = \bigcup_{\beta<\lambda} L_\beta, \text{ for limit } \lambda$$
$$L = \bigcup_{\alpha \in \mathrm{ON}} L_\alpha$$

Some facts about L:

Definition 0.2. An inner model of ZF(C) is a transitive class that contains all ordinals and satisfies the axiom of ZF(C).

Fact 0.3.
1. L is transitive
2. for all $\alpha \in \mathrm{ON}$, $L_\alpha \cap \mathrm{ON} = \alpha$. So L contains all the ordinals.
3. L is the smallest inner model of ZF. This means that if M is an inner model, then L^M, the class of all constructible sets in M, is just L. So $L \subseteq M$
4. $L \models ZFC + GCH$

Remark 0.4. One thing to notice about L as a cumulative hierarchy is that it grows very slowly compared to V. We don't know whether the two constructions eventually coincide or not. The hypothesis that they do is abbreviated $V = L$. It's not hard to see that $ZF + V = L \vdash AC + GCH$, since L models choice and GCH. Why aren't we inclined to accept $V = L$ as an axiom? Intuitively, one might think $V = L$ is too restrictive in some ways. We will see below that a theorem of Dana Scott demonstrates a concrete example of $V = L$'s limitation.

Theorem 0.5. (Mostowski Collapse) Let (X, E) be a (possibly proper class) structure, where $E \subseteq X \times X$ satisfies the following conditions:
1. (X, E) is extensional, i.e., $\forall a, b \in X\ a = b \iff (\forall c \in X\ cEa \iff cEb)$
2. E is well-founded
3. E is set-like, i.e., for each $a \in X$, the class $\mathrm{ext}(a) = \{b \in X \mid bEa\}$ is a set.

Then there is a unique transitive M and a unique isomorphism $\pi : (X, E) \to (M, \in)$

Note. π is going to look a lot like what we use in the trick of turning partial orders into ordering by subsets. Explicitly for each $a \in X$, $\pi(a) = \{\pi(b) \mid bEa\}$. This is well-defined because E is assumed to be well founded.

Transitive models of $ZF(C)$ look a lot like V. The following facts will make this claim more precise.

Definition 0.6. A formula φ is said to be absolute for a transitive model M iff,

$$\forall x_1, ..., x_n \in M \ \varphi^M(x_1, ..., x_n) \leftrightarrow \varphi(x_1, ..., x_n)$$

where φ^M denotes φ with quantifiers restricted to M. φ is said to be upward-absolute iff $\forall x_1, ..., x_n \in M \ \varphi^M(x_1, ..., x_n) \rightarrow \varphi(x_1, ..., x_n)$; downward absolute iff $\forall x_1, ..., x_n \in M \ \varphi(x_1, ..., x_n) \rightarrow \varphi^M(x_1, ..., x_n)$

Fact 0.7.
1. Δ_0 formulas are absolute for all transitive classes (or absolute for short)
2. Σ_1 formulas are upward-absolute.
3. Π_1 formulas are downward-absolute.
4. Δ_1 formulas are absolute.

Definition 0.8. the rank of a set x is rank(x) = the least α such that $x \subseteq V_\alpha$

Fact 0.9. This definition of rank has several properties that we will use. For example:
1. if $x \in y$, then rank(x)<rank(y)
2. if $x \subseteq y$, then rank(x)≤rank(y)
3. if $x \in V_\kappa$ for some limit ordinal κ, then rank(x) $< \kappa$

1 Measurable Cardinals to Elementary Embeddings

Recall:

Definition 1.1. (Ultrapower)
Given a structure M, and an ultrafilter U on a set S, we define the following relations on the product $\Pi_{i \in S} M = \{f \mid S \to M\}$:

$$f =^* g \iff \{i \in S \mid f(i) = g(i)\} \in U$$
$$f \in^* g \iff \{i \in S \mid f(i) \in g(i)\} \in U$$

We define equivalence classes of $\Pi_{i \in S} M$ modulo $=^*$:

$$[f] = \{g \mid f =^* g \wedge \underline{\forall h(h =^* f \rightarrow \text{rank}(g) \leq \text{rank}(h)}\}$$

(Note: the underlined clause is called "Scott's trick". This ensures the $[f]$'s are sets.)
The *ultrapower* of M is the structure of these equivalence classes $\text{Ult}_U(M) = \Pi_{i \in S} M/U = \{[f] : f \in \prod_{i \in X} M\}$. Presently we are concerned with the case where M is the von Neumann universe V. we write Ult for $\text{Ult}_U(V)$

Remark 1.2. Intuitively, we are "stringing" S-many copies of M, and we examine the representative properties of each thread $f : S \to V$. The sense of "representative" is made precise by setting φ to be representative of f iff $\{i \in S \mid \varphi(f(x))\} \in U$. Since filters pick out "large" subsets, this could be thought of as saying φ is representative of f iff the elements of its range that satisfy φ constitute a majority in S. (Side note: In social choice theory, non-principal ultrafilters are used to define a rule (called a social welfare function) for aggregating the preferences of infinitely many individuals.)

Fact 1.3. (Łoś)
$\mathrm{Ult}_U(M)$ is elementarily equivalent to M. This is just a special case of Łoś' theorem.

Definition 1.4. An elementary embedding is a function $j : M \to N$ such that for all $\vec{x} \in M$ and for all formulas $\varphi(\vec{x})$: $M \vDash \varphi(\vec{x}) \leftrightarrow N \vDash \varphi(j(\vec{x}))$

Claim 1.5. The function $j_U : V \to \mathrm{Ult}$ befined by $j_U(a) = [c_a]$ is an elementary embedding, where for each $a \in V$, $[c_a]$ is the equivalence class of the constant functions $c_a(i) = a$ which map every index i to a. This embedding is called the canonical embedding.

Proof. Let $a \in V$, then by Łoś' theorem, $V \vDash \varphi[a]$ iff $\{i \in S \mid V_i \vDash \varphi[a]\} \in U$ iff $\mathrm{Ult} \vDash \varphi[c_a]$ iff Ult $\vDash \varphi([j(a)])$. ☐

We know from the above that Ult is a proper class model of set theory. We might ask when/how similar it is to V. A partial answer is given by the following lemma.

Lemma 1.6. if U is a $\sigma-$complete ultrafilter on some S, then (Ult, $\in^*$) is well-founded.

Note. Since Ult and V are elementarily equivalent, Ult $\vDash$ "(Ult, $\in^*$) is well-founded". What this lemma shows is that $V \vDash$ "(Ult, $\in^*$) is well-founded"

Proof. Let U be a $\sigma-$complete ultrafilter on X, we show that there is no infinite descending $\in^*$-sequence in Ult.

Suppose for contradiction that $\{f_i\}_{i\in\omega}$ is such a sequence. We define, for each n, the set X_n to be $X_n = \{x \in S \mid f_{n+1}(x) \in f_n(x)\}$. Note that each X_n is nonempty because $f_{n+1} \in^* f_n$ implies, by definition, that $\{i \in S : f_{n+1}(i) \in f_n(i)\} \in U$.

By $\sigma-$completeness, $X = \bigcap_{n\in\omega} X_n \in U$. In particular, X is not empty. Fix some arbitrary member $x \in X$, this x will witness an infinite descending $\in$-chain $f_0(x) \ni f_1(x) \ni \ldots$, which contradicts the axiom of regularity. ☐

Lemma 1.7. Ult is set-like.

Proof. fix some f in Ult. If $g \in^* f$, we can find some $h =^* g$ with rank(h)≤rank(f). Let h be defined by $h(x) = g(x)$ if $g(x) \in f(x)$; $h(x) = \emptyset$ otherwise. Hence $h =^* g$. But also the rank of the range of h is bounded by that of f, so h is a set. That is, $[h]$ is a set and is of lower rank than f. So ext(f) is a set. ☐

By Mostowski Collapse, there is a unique $\pi : (M, \in) \cong (\mathrm{Ult}, \in^*)$. For notational convenience, whenever U is σ-complete, we use the symbol Ult and M to denote the transitive collapse and identify $[f]$ with its image $\pi([f])$, unless noted otherwise. Occassionally we would want to look at the uncollapsed ultrapower; we reserve the symbol $\mathrm{Ult}_U(V)$ for that.

Thus if U is a σ-complete (i.e., ω_1-complete) ultrafilter on S, then $M = \mathrm{Ult}$ is an inner model of set theory and $j = \pi \circ j_U$ is an elementary embedding $j : V \to M$.

Remark 1.8. j has the following properties:
1. if α is an ordinal, $j(a)$ is also an ordinal; moreover, $\alpha < \beta$ implies $j(\alpha) < j(\beta)$. So $\alpha \le j(\alpha)$
2. $j(\alpha + 1) = j(\alpha) + 1$, and $j(n) = n$ for all $n \in \omega$.
3. $j(\omega) = \omega$.

Proof. 1. the first two claims by elementarity. The third claim: suppose not, let β be least such that $j(\beta) < \beta$. Then $j(j(\beta)) < j(\beta) < \beta$, contradicting the minimality of β.
2. notice that $\alpha + 1 = \alpha \cup \{\alpha\}$, so by elementarity, $j(\alpha+1) = j(\alpha) \cup \{j(\alpha)\} = j(\alpha) + 1$. The second

claim is obvious.
3. we prove the more general claim that if U is κ-complete, then $j(\alpha) = \alpha$ for all $\alpha < \kappa$. Recall that $j(\alpha) \geq \alpha$ for all α. Let $\alpha < \kappa$ and let $\beta < j(\alpha)$. We first show that $\beta = j(\gamma)$ for some $\gamma < \alpha$.

Since $\beta \in j(\alpha)$, there must be some $[f] \in \mathrm{Ult}_U(V)$ (note that we are talking about the uncollapsed ultrapower) such that $\beta = \pi([f]) \in \pi \circ j_U(\alpha)$. So for this $[f]$, we have $\mathrm{Ult}_U(V) \vDash [f] \in^* [c_\alpha]$, which means that $\{i \in S \mid f(i) \in c_\alpha(i) = \alpha\} \in U$. We shall show that there is some $\gamma < \alpha$ such that $[f] = [c_\gamma]$.

For each $\gamma < \alpha$, let $A_\gamma = \{i \in S \mid f(i) \neq \gamma\}$. Then note that $\bigcap_{\gamma<\alpha} A_\gamma \notin U$ (why?). But since U is κ-complete, this can only mean that one of the A_γ is not in U. But then the ultra condition implies that $S \smallsetminus A_\gamma = \{i \in S \mid f(i) = \gamma\} \in U$. So this tells us that $[f] = [c_\gamma]$. Therefore, $\beta = \pi([f]) = \pi([c_\gamma]) = \pi \circ j_U(\gamma)$.

So $j(\alpha) = \{j(\gamma) \mid \gamma < \alpha\}$. By induction on $\alpha < \kappa$, we see that $j(\alpha) = \alpha$. □

We might ask whether j is the identity function on the ordinals. The following series of claims will show that it is not.

Claim 1.9. If U is a κ-complete, nonprincipal ultrafilter on κ, then every bounded subset B of κ is will not be in U. This is because every $\kappa \smallsetminus \{\alpha\}, \alpha \in \kappa$ is in U. So $\kappa \smallsetminus B = \bigcap_{\beta\in B}(\kappa \smallsetminus \{\beta\})$ is in U .

Definition 1.10. The diagonal function $d : \kappa \to \kappa$ is the function defined by $d(\alpha) = \alpha$, for all $\alpha \in \kappa$.

Claim 1.11. For every $\gamma < \kappa$, $\{\alpha \in \kappa \mid d(\alpha) > \gamma\} \in U$. This follows from claim 1.9. (picture?)

Claim 1.12. $[d] > \gamma$ for all $\gamma < \kappa$, and therefore $\kappa \leq [d]$. However, since we have $[d] < j(\kappa)$, it follows that $\kappa < j(\kappa)$.

This shows that if there is a measurable cardinal, then j is a nontrivial elementary embedding (i.e., j is not the identity). With this, we can show our first main theorem.

Theorem 1.13. (Scott) If there is a measurable cardinal, then $V \neq L$

Proof. Suppose there is a measurable cardinal; let κ be the least measurable cardinal and let U witness this. And let $j : V \to M$ be the corresponding elementary embedding.

Suppose for contradiction that $V = L$. Then since M is an inner model and L is the smallest inner model, $V = M = L$. But $M \vDash j(\kappa)$ is the least measurable cardinal, by elementarity. This contradicts the fact that $j(\kappa) > \kappa$. □

Remark 1.14. Note that 1.13 does not show that κ is not in L (recall that L contains all the ordinals), just that the witnessing ultrafilter is not in L. More precisely, inaccessibility and Mahloness, etc. are Π_1 properties whereas measurability is Σ_2. Heuristically, inaccessible cardinals are in some sense about how high up the hierarchy goes, whereas measurable cardinals require that the universe should be "wide" as well. This provides an intuition as to why they are called large large-cardinals.

2 Elementary Embeddings to Large Cardinals

So far, we've shown that if there is a measurable cardinal, then there is a nontrivial elementary embedding $j : V \to M$. In this section we shall see that the converse also holds.

Theorem 2.1. If $j : V \to M$ is a nontrivial elementary embedding of the universe, then there exists a measurable cardinal.

Proof. Fix such an embedding $j : V \to M$. We show that there is an ordinal α such that $j(\alpha) \neq \alpha$. Suppose not, then by induction on rank we show that j is trivial (the identity function). Suppose $j(x) = x$ for all x with rank less than β. We show that if rank(y)=β, then $j(y) = y$. But if rank(y)=β, then rank($j(y)$) $= j(\beta) = \beta$. If $x \in j(y)$, then rank(x)$< \beta$, so $j(x) \in j(y)$; therefore $x \in y$. Conversely, if $x \in y$, then rank(x)$< \beta$, so $j(x) \in y$. Therefore $j(y) = y$ for all y. This contradict the nontriviality of j. So there is a least κ such that $j(\kappa) > \kappa$.

We note that formulas such as "$x = n$" for all finite ordinals n and $x = \omega$ are Δ_0 and therefore absolute (recall fact 0.7). This means that $\forall x \in M \ M \vDash x = n$ iff $x = n$, and $M \vDash x = \omega$ iff $x = \omega$ This shows that $j \restriction \omega + 1 = id \restriction \omega + 1$. Therefore $\kappa > \omega$. We now prove that κ is in fact a measurable cardinal.

By definition, it suffices to show that there is a κ-complete nonprincipal ultrafilter on κ. We let $D \subseteq \mathcal{P}(\kappa)$ be defined as:

$$\text{for all X} \subseteq \kappa : X \in D \iff \kappa \in j(X)$$

We show that D is a κ-complete nonprincipal ultrafilter.

First, D is a filter: $\kappa \in j(\kappa)$ is just $\kappa < j(\kappa)$. $\emptyset \notin D$ because $j(\emptyset) = \emptyset$. Upward closure: $X \subseteq Y$ implies $j(X) \subseteq j(Y)$ by elementarity. So if $X \subseteq Y \in \mathcal{P}(\kappa)$ is such that $\kappa \in j(X)$, then we have $\kappa \in j(Y) \supseteq j(X)$. Downward directedness: Note that $j(X \cap Y) = j(X) \cap j(Y)$, so if $\kappa \in j(X)$ and $\kappa \in j(Y)$, then $\kappa \in j(X \cap Y)$. Ultra: by elementarity, if $Y \subseteq \kappa$, then $\kappa \in j(\kappa) = j(Y) \cup j(\kappa \setminus Y)$. So either Y or $\kappa \setminus Y$ will be in D.

D is nonprincipal: for every $\alpha < \kappa$, we have that $j(\{\alpha\}) = \{j(\alpha)\} = \{\alpha\}$, therefore D doesn't contain any singleton subsets of κ, so it's nonprincipal. (This follows from the fact that principal ultrafilters must contain a singleton).

D is κ-complete: let $\gamma < \kappa$ and $f : \gamma \to U$, then we want to show that $\kappa \in j(\bigcap_{\alpha<\gamma} f(\alpha)) = \bigcap_{\alpha<j(\gamma)} j(f)(\alpha)$. So <u>$f$ is a sequence of length γ of subsets of κ, where each αth term is $f(\alpha)$</u>. Note that the underlined sentence is Δ_0 and therefore absolute.

So, in M as well as in V, we have that $j(f)$ is a sequence of length $j(\gamma)$ of subsets of $j(\kappa)$, where each $j(\alpha)$th term is $j(f(\alpha))$. But since $j(\alpha) = \alpha$ for all $\alpha \leq \gamma < \kappa$, we have that $j(f)(j(\alpha)) = j(f)(\alpha) = j(f(\alpha))$. And so $\bigcap_{\alpha<j(\gamma)} j(f)(\alpha) = j(\bigcap_{\alpha<\gamma} f(\alpha)) = \bigcap_{\alpha<\gamma} j(f(\alpha))$. Note that $\kappa \in j(f(\alpha))$ for each such α by definition. Hence $\kappa \in j(\bigcap_{\alpha<\gamma} f(\alpha))$ $\square$

So we've just shown that the assertion that there exists a measurable cardinal is equivalent to the assertion that there exists a nontrivial elementary embedding from the universe to some inner model. Many large large-cardinal axioms we are going to encounter will come in this format. Finally, we show some more properties about nontrivial elementary embeddings from the universe to the collapsed ultrapower from measurable cardinals.

Lemma 2.2. let κ be a measurable cardinal and U be the witnessing ultrafilter. Let j be the corresponding elementary embedding from V to the collapsed ultrapower M

1. $j \restriction V_\kappa = id \restriction V_\kappa$
2. $V_{\kappa+1} = (V_{\kappa+1})^M$
3. ${}^\kappa M \subseteq M$
4. $U \notin M$, and so $V_{\kappa+2} \nsubseteq M$

Proof. 1. Since j is not trivial, we can fix x of least rank such that $j(x) \neq x$ (recall 0.9). We show that rank(x)$\geq \kappa$.

First, note that if $y \in x$, then rank(y) <rank(x), and so by our assumption, $y = j(y) \in j(x)$. This shows that $x \subseteq j(x)$ and also rank(x)≤rank$(j(x))$. So there must be some $z \in j(x) \setminus x$.

Let rank(x)=$\delta \leq$ rank$(j(x))$. We show that rank$(j(x))$ cannot be δ. Suppose for contradiction that rank$(j(x)) = \delta$. But then $z \in j(x) \setminus x$ will have rank strictly less than δ. This means that $z = j(z) \in j(x)$. By elementarity, $z \in x$, which is a contradiction.

Therefore, rank$(j(x)) > \delta$. Since rank$(x) = \delta$, we have rank$(j(x)) = j(\delta) > \delta$. But least such δ is κ. So rank$(x) \geq \kappa$. In particular, $x \notin V_\kappa$.

2.if $x \subseteq V_\kappa$, then $x = (j(x) \cap V_\kappa)$. Therefore $V_{\kappa+1} = V^M_{\kappa+1}$

3. suppose $\langle [f_\alpha] : \alpha < \kappa \rangle$ is a sequence of members of M. We want to find a $g : \kappa \to V$ such that $[g] = \langle [f_\alpha] : \alpha < \kappa \rangle$. So let $h : \kappa \to \kappa$ be such that $[h] = \kappa$. For each $\gamma < \kappa$, let $g(\gamma)$ be a function from $h(\gamma)$ to V, defined by $(g(\gamma))(\alpha) = f_\alpha(\gamma)$. Therefore, by Łoś' theorem, $[g]$ is a function from κ to V such that $[g](\alpha) = f(\alpha)$.

4. First, note that ${}^\kappa\kappa = ({}^\kappa\kappa)^M \in M$ by 1 and 2. Suppose for contradiction that $U \in M$, then the map g that maps $f \in {}^\kappa\kappa$ to $[f] \in j(\kappa)$ will be in M. But this would mean that $M \vDash \mid j(\kappa) \mid \leq \kappa^\kappa = 2^\kappa$, which contradicts the fact that $M \vDash j(\kappa)$ is measurable (and so inaccessible). □

Remark 2.3. These facts place certain limitations on how closely M resembles V in our setting. We may assert the existence of elementary embeddings that surpass these limitations (for example, an embedding where $V_{\kappa+2} \subseteq M$). These assertions will result in stronger large cardinal axioms. Naturally, we may ask how close M can be to V and how "correct" j will be. For example, Kunen showed that the requirement that $M = V$ is inconsistent with ZFC. Whether there exists nontrivial elementary embedding from V to V in ZF is still an open question.