A LOCALLY COUNTABLE COANALYTIC GRAPH NOT GENERATED BY COUNTABLY MANY REAL-ORDINAL DEFINABLE FUNCTIONS

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Abstract. Let κ be inaccessible in L, and let

$$\mathfrak{N} = L[G], \qquad G \subseteq \mathrm{Coll}(\omega, {<}\kappa),$$

be the Levy collapse extension. We construct a lightface Π^1_1 equivalence relation on the reals whose classes are countable in $\mathfrak{N}$, while its irreflexive part is not generated by any countable family of real-ordinal definable partial functions. This gives an affirmative answer to a question of Kanovei and Lyubetsky. The construction uses the relativized largest thin set $C_1(z)$ to obtain, uniformly in z, a coanalytic subgroup $H_z \subseteq L[z]$ which is uncountable in $L[z]$. In the collapse extension the cosets of H_z are countable. Given a putative countable ROD generating family, an enumeration can be made $\mathrm{OD}(z)$ for one real z, and the argument is localized to a Cohen extension of $L[z]$, where a generic coset of H_z is an uncountable clique. A ccc antichain argument rules out generation of this clique.

1. Introduction

Let X be a set and let $\mathcal{G} \subseteq X^2$ be an irreflexive symmetric relation. A family $\mathcal{F}$ of partial functions from X to X *generates* $\mathcal{G}$ if, for distinct $x, y \in X$,

$$x \mathrel{\mathcal{G}} y \iff \exists f \in \mathcal{F} \bigl(f(x) = y \ \vee\ f(y) = x\bigr).$$

An equation involving $f(x)$ includes the assertion that $x \in \mathrm{dom}(f)$. A set is *real-ordinal definable* (ROD) if it is definable from a real and finitely many ordinals; a partial function is ROD if its graph is. A countable family of ROD functions will always mean a countable family whose members are individually ROD; no definability of the family itself is assumed. The Luzin–Novikov theorem implies that every locally countable Borel graph is generated by countably many partial Borel functions.

Kanovei and Lyubetsky asked whether, in the Solovay model, there is a locally countable Π^1_1 graph which is not generated by a countable family of ROD functions [3, Problem 1]. We note that the term "Solovay model" in their paper refers to the full collapse extension

$$\mathfrak{N} = L[G], \qquad G \subseteq \mathrm{Coll}(\omega, {<}\kappa),$$

2020 *Mathematics Subject Classification.* Primary 03E15; Secondary 03E35, 05C63.

Key words and phrases. coanalytic graph, real-ordinal definability, Solovay model, Cohen forcing.

where κ is inaccessible in L [3, Definition 10], instead of the choiceless inner model that is also often called the Solovay model.

We answer their question affirmatively. In fact, the graph can be taken to be the irreflexive part of a lightface coanalytic equivalence relation.

Theorem 1.1. *In $\mathfrak{N}$ there is a lightface Π^1_1 equivalence relation E on the reals such that every E-class is countable, but the graph*

$$\mathcal{G} = E \setminus \{(x, x) : x \in 2^\omega\}$$

is not generated by any countable family of ROD partial functions.

The proof has two components. Its descriptive-set-theoretic component is based on a construction of Friedman and Törnquist, using ideas of Miller and Conley, recorded by Chan [1, Theorem 11.1]. In that construction, an uncountable thin coanalytic set is placed inside a linearly independent perfect set; the subgroup it generates is coanalytic, and the corresponding coset relation has uncountable thin classes. Here we apply the same finite-span mechanism uniformly to the relativized largest thin sets $C_1(z)$. For each real z this yields a coanalytic subgroup H_z of the Cantor group which is contained in $L[z]$ and uncountable in $L[z]$. The equivalence classes over the parameter z are the cosets of H_z. They become countable in $\mathfrak{N}$ because $L[z] \cap 2^\omega$ is countable there.

In the forcing component, we use an antichain argument to rule out countable ROD generation. Suppose that a countable ROD family generated the graph. In $\mathfrak{N}$ the family can be defined from one real z_0 and ordinal parameters. We choose in $\mathfrak{N}$ a Cohen real a over $L[z_0]$ and restrict the family to $L[z_0, a]$. The coset

$$\{\langle z_0, a + h\rangle : h \in H_{z_0}\}$$

is an uncountable clique in this intermediate extension. Its elements are obtained from one another by ground-model translations and all generate the same Cohen extension. Every edge of the graph has the additional property that either endpoint belongs to the constructible universe generated by the other. Weak homogeneity and the ccc then imply that one definable function has only countably many points of the clique in any fixed inverse fiber. This is incompatible with countable generation.

2. The coanalytic equivalence relation

For a real r, let ω_1^r denote supremum of r-recursive ordinals. For reals z, x, write $z \oplus x$ for their Turing join. We regard 2^ω as the Cantor group with coordinatewise addition modulo 2, denoted by $+$. A set of reals is *thin* if it contains no nonempty perfect subset.

For $z \in 2^\omega$, let

$$C_1(z) = \{x \in 2^\omega : x \in L_{\omega_1^{z \oplus x}}[z]\}. \tag{1}$$

We shall use the following properties of this set: the relation $x \in C_1(z)$ is lightface Π^1_1 in (z, x), $C_1(z) \subseteq L[z]$, $C_1(z)$ is the largest $\Pi^1_1(z)$ thin set, and that $L[z]$ regards $C_1(z)$ as uncountable.. For details, see [2, Theorem 4.3.3].

We next place $C_1(z)$ inside a recursively presented linearly independent perfect set. Fix a recursive bijection $e : \omega \to 2^{<\omega}$, and define $\vartheta : 2^\omega \to 2^\omega$ by

$$\vartheta(x)(n) = 1 \iff e(n) \subseteq x.$$

Let $D = \vartheta[2^\omega]$. Then D is a recursive perfect set and ϑ is a recursive homeomorphism of 2^ω onto D. Moreover, D is linearly independent over $\mathbb{F}_2$. Indeed, given distinct $x_0, \dots, x_{k-1}$, choose m such that their initial segments of length m are pairwise distinct. At the coordinate corresponding to $x_i \restriction m$, the real $\vartheta(x_i)$ has value 1 and all the other $\vartheta(x_j)$ have value 0.

Put

$$P_z = \vartheta[C_1(z)], \qquad H_z = \operatorname{span}_{\mathbb{F}_2}(P_z).$$

Thus H_z is the set of finite sums of members of P_z, including the empty sum 0.

Proposition 2.1. *The relation $h \in H_z$ is lightface Π^1_1. For every $z \in 2^\omega$, we have $H_z \subseteq L[z]$, and $L[z]$ satisfies that H_z is uncountable. If $V \subseteq V[K]$ is a forcing extension and $z \in V$, then H_z has the same members in V and $V[K]$.*

Proof. Let $\operatorname{span}(D)$ be the set of finite sums of members of D. By the linear independence of D, every $h \in \operatorname{span}(D)$ has a unique finite support $\operatorname{supp}(h) \subseteq D$. The relation saying that a strictly lexicographically increasing finite sequence from D has sum h is lightface Borel and has at most one such sequence for each h. The effective Luzin–Novikov theorem therefore implies that $\operatorname{span}(D)$ is lightface Borel and that the finite-support map $h \mapsto \operatorname{supp}(h)$ is lightface Borel on its domain; see, for example, [4, Section 4D]. This is the Cantor-group version of the finite-support argument used in [1, Theorem 11.1].

For $h \in 2^\omega$ we now have

$$h \in H_z \iff h \in \operatorname{span}(D) \ \wedge\ \operatorname{supp}(h) \subseteq P_z.$$

Membership in P_z is uniformly $\Pi^1_1(z)$, and the support is finite and Borel-decodable. Hence the displayed condition is uniformly $\Pi^1_1(z)$.

Since $C_1(z) \subseteq L[z]$, also $P_z \subseteq L[z]$, and therefore $H_z \subseteq L[z]$. The set P_z is uncountable in $L[z]$ and is contained in H_z, so H_z is uncountable there.

Finally suppose that $V \subseteq V[K]$ and $z \in V$. For any real $h \in V$, the statement $h \in H_z$ is Π^1_1 and hence is absolute between V and $V[K]$. This shows that every member of H_z computed in V remains a member in $V[K]$. Conversely, the argument above, carried out in $V[K]$, shows that every member of H_z computed there belongs to $L[z]$. Forcing does not change $L[z]$, so such an h already belongs to V, and absoluteness then puts it in the ground-model computation of H_z. $\square$

Define an equivalence relation on $2^\omega \times 2^\omega$ by

$$\langle z, x\rangle \ E\ \langle w, y\rangle \iff z = w \ \wedge\ x + y \in H_z. \tag{2}$$

Proposition 2.2. *The relation E is a lightface Π^1_1 equivalence relation. Its classes are*

$$[\langle z,x\rangle]_E = \{\langle z, x+h\rangle : h \in H_z\}.$$

Moreover, if distinct u, v are E-equivalent, then

$$u \in L[v] \qquad \text{and} \qquad v \in L[u]. \tag{3}$$

Proof. The complexity follows from Proposition 2.1. Reflexivity, symmetry, and transitivity follow from the fact that each H_z is a subgroup.

Write $u = \langle z, x\rangle$ and $v = \langle z, y\rangle$, and suppose that $x + y = h \in H_z$. By Proposition 2.1, $h \in L[z]$. The real z is recursively recoverable from v, so $L[z] \subseteq L[v]$. It follows that $x = y + h \in L[v]$, and hence $u \in L[v]$. The other inclusion is symmetric. $\square$

Let $\mathcal{G} = E \setminus \Delta$ be the irreflexive part of E. It is a lightface Π^1_1 graph.

3. A c.c.c. argument

We first isolate the elementary ccc fact used in the proof. Fix a parameter-free definable class function

$$d : \mathrm{Ord} \times 2^\omega \longrightarrow 2^\omega$$

such that, for every real y,

$$\{d(\xi, y) : \xi \in \mathrm{Ord}\} = L[y] \cap 2^\omega.$$

One may take $d(\xi, y)$ to be the ξth real in the canonical well-order of $L[y]$, allowing repetitions and a default value.

Lemma 3.1. *Let M be a transitive model of ZFC, let $\mathbb{P} \in M$ be a ccc forcing notion, and let $\dot{x}, \dot{y} \in M$ be names for reals such that*

$$\mathbf{1}_{\mathbb{P}} \Vdash_M \dot{x} \in L[\dot{y}].$$

Then there is a maximal antichain $A \in M$ and ordinals ξ_p, for $p \in A$, such that

$$p \Vdash_M \dot{x} = d(\check{\xi}_p, \dot{y}) \qquad (p \in A).$$

Consequently, suppose that W is an outer model containing a family $\{G_i : i \in I\}$ of M-generic filters such that $M[G_i] = W$ for every $i \in I$. Put

$$x_i = \dot{x}^{G_i}, \qquad y_i = \dot{y}^{G_i}.$$

For every fixed real $b \in W$, the set

$$\{x_i : y_i = b\}$$

is countable in W.

Proof. Given $q \in \mathbb{P}$, the hypothesis implies

$$q \Vdash_M \exists \xi \in \mathrm{Ord}\, \big(\dot{x} = d(\xi, \dot{y})\big).$$

By the maximal principle and a further strengthening, there are $p \le q$ and an ordinal ξ such that $p \Vdash_M \dot{x} = d(\check{\xi}, \dot{y})$. Thus the set of such conditions is dense. Choose a maximal antichain A contained in this dense set and, for each $p \in A$, choose a corresponding ordinal ξ_p. The antichain is countable because $\mathbb{P}$ is ccc.

For the final assertion, let p_i be the unique member of $A \cap G_i$. If $y_i = b$, then in W

$$x_i = d(\xi_{p_i}, b).$$

Therefore

$$\{x_i : y_i = b\} \subseteq \{d(\xi_p, b) : p \in A\},$$

and the set on the right is countable. □

Fix $z \in 2^\omega$, put $M = L[z]$, and let a be Cohen-generic over M. Let

$$W = M[a].$$

For $h \in H_z$ as computed in M, put

$$a_h = a + h, \qquad u_h = \langle z, a_h \rangle.$$

Translation by h is an automorphism of Cohen forcing, so a_h is Cohen-generic over M and

$$M[a_h] = M[a] = W. \tag{4}$$

By Proposition 2.1, the group H_z is the same in M and W. It is uncountable in M, and Cohen forcing preserves its uncountability. Therefore

$$X = \{u_h : h \in H_z\} \tag{5}$$

is an uncountable set in W. It is a clique in $\mathcal{G}$: for distinct h, k,

$$(a + h) + (a + k) = h + k \in H_z.$$

Proposition 3.2. *In $W = L[z, a]$, the graph $\mathcal{G}$ is not generated by any countable family of partial functions each individually* $\mathrm{OD}^W(z)$.

Proof. Suppose that $\{f_n : n < \omega\}$ is such a generating family. Fix $u_* = u_{h_*} \in X$. For each n, define a total function $\bar{f}_n$ by

$$\bar{f}_n(u) = \begin{cases} f_n(u), & u \in \mathrm{dom}(f_n), \\ u, & u \notin \mathrm{dom}(f_n). \end{cases}$$

This function is $\mathrm{OD}^W(z)$. Since the original family generates $\mathcal{G}$, every non-diagonal pair $(u, f_n(u))$ is an edge of $\mathcal{G}$. Proposition 2.2 therefore gives

$$W \models \forall u \in 2^\omega \big(u \in L[\bar{f}_n(u)]\big). \tag{6}$$

Choose a formula $\varphi_n(u, v, z, \vec{\alpha})$, with a finite tuple of ordinal parameters $\vec{\alpha}$, which defines the graph of $\bar{f}_n$ in W. Let $\Psi_n(z, \vec{\alpha})$ be the assertion that φ_n defines a total function on 2^ω and that

$$\forall u, v \in 2^\omega \big(\varphi_n(u, v, z, \vec{\alpha}) \longrightarrow u \in L[v]\big).$$

The model W satisfies $\Psi_n(z, \vec{\alpha})$, so some condition in the original generic filter forces it. Cohen forcing is weakly homogeneous and all the parameters belong to M. Hence

$$\mathbf{1} \Vdash_M \Psi_n(\check{z}, \check{\vec{\alpha}}).$$

Let $\dot{a}$ be the canonical name for the Cohen real and let

$$\dot{u} = \langle \check{z}, \dot{a} \rangle.$$

By the maximal principle there is a name $\dot{y}_n \in M$ such that

$$\mathbf{1} \Vdash_M \varphi_n(\dot{u}, \dot{y}_n, \check{z}, \check{\vec{\alpha}}).$$

Consequently,

$$\mathbf{1} \Vdash_M \dot{u} \in L[\dot{y}_n].$$

For $h \in H_z$, let G_h be the translate of the original Cohen-generic filter whose generic real is a_h. By (4), every $M[G_h]$ is the same universe W. Since the formula $\varphi_n(u, v, z, \vec{\alpha})$ defines $\bar{f}_n$ in this universe,

$$\dot{u}^{G_h} = u_h, \qquad \dot{y}_n^{G_h} = \bar{f}_n(u_h).$$

Applying Lemma 3.1 to $\dot{u}$ and $\dot{y}_n$ shows that, for every fixed $v \in W$, the set

$$\{u_h \in X : \bar{f}_n(u_h) = v\}$$

is countable. In particular, the incoming set

$$I_n = \{u_h \in X : f_n(u_h) = u_*\}$$

is countable.

The outgoing value $f_n(u_*)$, when defined, accounts for at most one member of X. Since X is a clique generated by the functions f_n, every $u \in X \setminus \{u_*\}$ belongs, for some n, either to I_n or to the singleton $\{f_n(u_*)\}$. Thus X is countable, contradicting (5). □

4. The Levy collapse extension

Let κ be inaccessible in L, and let

$$\mathfrak{N} = L[G], \qquad G \subseteq \mathrm{Coll}(\omega, {<}\kappa)$$

be a generic extension of L. We use the following standard properties of this model, collected in [3, Proposition 11]:

(1) every sequence of ROD sets is itself ROD;
(2) if $A \subseteq 2^\omega$ is $\mathrm{OD}(z)$, then for every real a the set $A \cap L[z, a]$ belongs to $L[z, a]$ and is $\mathrm{OD}(z)$ there;
(3) $L[z] \cap 2^\omega$ is countable in $\mathfrak{N}$ for every real z.

We now prove the main theorem.

Proof of Theorem 1.1. The relation E defined by (2) is lightface Π^1_1 by Proposition 2.2. Fix $\langle z, x \rangle \in 2^\omega \times 2^\omega$. Its E-class is contained in

$$\{\langle z, x + h \rangle : h \in L[z] \cap 2^\omega\},$$

which is countable in $\mathfrak{N}$. Hence E is a countable equivalence relation in $\mathfrak{N}$, and $\mathcal{G} = E \setminus \Delta$ is a locally countable lightface Π^1_1 graph.

Suppose toward a contradiction that a countable family of ROD partial functions generates $\mathcal{G}$ in $\mathfrak{N}$. Since $\mathfrak{N}$ satisfies ZFC, choose an enumeration $\langle f_n : n < \omega \rangle$. By property (1), the sequence is ROD. Thus there is a real z_0

such that the sequence is $\mathrm{OD}(z_0)$, where ordinal parameters are allowed in this notation.

There is in $\mathfrak{N}$ a Cohen real a over $L[z_0]$. Indeed, every dense subset of the countable Cohen forcing which belongs to $L[z_0]$ is coded by a real of $L[z_0]$, and the set of such reals is countable in $\mathfrak{N}$ by property (3). Enumerating the dense sets and meeting them successively produces the required real.

Put $W = L[z_0, a]$. Using a fixed recursive coding of tuples by reals, code the graphs of the sequence by

$$A = \{\langle n, x, y\rangle : f_n(x) = y\}.$$

The set A is $\mathrm{OD}(z_0)$ in $\mathfrak{N}$. By property (2),

$$A' = A \cap W$$

belongs to W and is $\mathrm{OD}(z_0)$ there. Decode A' as a sequence of partial functions $\langle g_n : n < \omega\rangle$ on the reals of W. Each g_n is $\mathrm{OD}^W(z_0)$.

For reals $x, y \in W$, the statement $x \mathrel{\mathcal{G}} y$ is absolute between W and $\mathfrak{N}$ by Mostowski absoluteness, since $\mathcal{G}$ is lightface Π^1_1. The sequence $\langle g_n : n < \omega\rangle$ generates the graph computed in W. To see this, let $x, y \in W$ be adjacent. They are adjacent in $\mathfrak{N}$, so for some n either $f_n(x) = y$ or $f_n(y) = x$. The relevant coded triple belongs to A'. Conversely, if $x \neq y$ and $g_n(x) = y$, then $f_n(x) = y$ in $\mathfrak{N}$, so $x \mathrel{\mathcal{G}} y$ there and hence also in W; the same applies with x and y reversed.

This contradicts Proposition 3.2, applied with $z = z_0$. $\square$

Kanovei and Lyubetsky also consider ROD chromatic numbers. As in [3], $\chi_{\mathrm{ROD}}(\mathcal{G})$ is the least possible cardinality of the range of a ROD proper coloring from the vertex set to 2^ω.

Corollary 4.1. *In $\mathfrak{N}$, the graph $\mathcal{G}$ has ROD chromatic number*

$$\chi_{\mathrm{ROD}}(\mathcal{G}) = 2^{\aleph_0}.$$

Proof. Suppose that $c : 2^\omega \to 2^\omega$ is a ROD proper coloring of $\mathcal{G}$ with countable range C. For each $r \in C$, define a partial function f_r by

$$f_r(x) = y \iff x \mathrel{\mathcal{G}} y \ \wedge\ c(y) = r.$$

This relation is functional. Indeed, if y and y' are both neighbors of x, then x, y, y' lie in one E-class. If $y \neq y'$, then $y \mathrel{\mathcal{G}} y'$, and properness gives $c(y) \neq c(y')$.

Each f_r is ROD, using r and the parameters defining c, and the family $\{f_r : r \in C\}$ is countable. If $x \mathrel{\mathcal{G}} y$, then $f_{c(y)}(x) = y$. Hence this family generates $\mathcal{G}$, contrary to Theorem 1.1. Thus $\mathcal{G}$ has no countable ROD coloring.

The range of any ROD coloring is a ROD set of reals. In the Levy–Solovay extension, every uncountable ROD set of reals has a perfect subset and hence has size $2^{\aleph_0}$ [3, Proposition 11(v)]. The reverse inequality is given by the identity coloring. $\square$

5. Remarks

The dependence of H_z on the first coordinate is essential to the argument. In the Friedman–Törnquist construction recorded by Chan, a single constructibly coded coanalytic subgroup suffices for the question considered there. Here a fixed coanalytic subgroup contained in L becomes countable in $\mathfrak{N}$, and it is already countable in every intermediate model $L[z, a]$ whenever ω_1^L is countable in $L[z]$. Such a subgroup would not provide the uncountable clique used in Proposition 3.2. The uniform family obtained from the relativized largest thin sets $C_1(z)$ avoids this obstruction: H_z is uncountable in $L[z]$ for every real z. Once a purported generating family is coded by z_0, the fiber over z_0 supplies the required clique.

Kanovei and Lyubetsky propose a different graph in [3, Problem 1], obtained by lifting equi-constructibility through a coanalytic uniformization. In the present construction, the extra coordinate selects a subgroup, and the second coordinate carries a generic coset. This makes it possible to use translations of one Cohen real rather than canonizing the witnesses of a uniformization.

References

[1] W. Chan, *Equivalence relations which are Borel somewhere*, J. Symbolic Logic **82** (2017), no. 3, 893–930.

[2] C. T. Chong and L. Yu, *Recursion Theory: Computational Aspects of Definability*, De Gruyter Series in Logic and Its Applications, vol. 8, De Gruyter, Berlin/Boston, 2015.

[3] V. Kanovei and V. Lyubetsky, *Locally countable graphs of second projective class not generated by countably many projective functions*, arXiv:2605.03126v6 [math.LO], 2026.

[4] Y. N. Moschovakis, *Descriptive Set Theory*, 2nd ed., Mathematical Surveys and Monographs, vol. 155, American Mathematical Society, Providence, RI, 2009.