ABSOLUTELY Δ^1_2 BOREL SETS HAVE CONSTRUCTIBLE BOREL CODES ON COUNTABLE ORDINALS

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ABSTRACT. Carl, Schlicht, and Welch asked whether every absolutely Δ^1_2 Borel set has a Borel$^{(<\omega_1)}$-code in L, and whether the code can be chosen at the same level of the Borel hierarchy. We prove the stronger, level-preserving statement in ZFC, removing the inaccessible-in-L assumption from Stern's corollary. The proof applies Stern's separation theorem after forcing its indexing ordinal to become countable and then uses Shoenfield absoluteness to return to the ground model. The key point is that, for a fixed ordinary Borel code b, the existence of an equivalent constructible generalized code indexed by a countable ordinal and of a prescribed Borel class is Σ^1_2.

1. Introduction

Throughout, projective pointclasses are lightface unless a real parameter is displayed. We use the coding conventions of Carl–Schlicht–Welch [1, Section 2.2]. For a multiplicatively closed ordinal $\lambda \geq \omega$, a Borel$^{(\lambda)}$-code is obtained from an ordinary Borel code by replacing ω with λ; the notions of $\Sigma^{(\lambda)}_\alpha$- and $\Pi^{(\lambda)}_\alpha$-codes are defined accordingly. A $\Sigma^{(<\omega_1)}_\alpha$-code is a $\Sigma^{(\lambda)}_\alpha$-code for some multiplicatively closed λ with $\omega \leq \lambda < \omega_1$, and similarly for Π-codes. When λ is countable, a bijection from ω onto λ converts such a code into an ordinary Borel code without increasing its Borel class. Thus generalized codes on countable ordinals and ordinary Borel codes define the same sets in the ambient universe. The distinction is nevertheless important for inner models: a code in L may be indexed by an ordinal that is countable in V even though the corresponding enumeration, and hence the resulting ordinary code, does not belong to L.

A set $A \subseteq 2^\omega$ is *absolutely* Δ^1_2 if there are a Σ^1_2 formula $\varphi(x)$ and a Π^1_2 formula $\psi(x)$ such that

$$\{x \in 2^\omega : \varphi(x)\} = \{x \in 2^\omega : \psi(x)\}$$

in every set-forcing extension of V.

At the first projective level, Kechris, Marker, and Sami proved that every Π^1_1 Borel set has a Borel$^{(<\omega_1)}$-code in L [3]; see also [1, Proposition 5.3]. Carl–Schlicht–Welch posed as their main question whether the analogous conclusion holds for absolutely Δ^1_2 Borel sets. Stern [5] proved the conclusion under the additional hypothesis that ω_1 is inaccessible in L. Carl–Schlicht–Welch note that the method of Louveau's separation theorem [4] does not apply here [1, Question 6.1]. They also ask for the level-by-level conclusion: if the set belongs to a given Borel class, can the constructible generalized code be chosen in the same class? They observe that such

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a result would strengthen several classical theorems of descriptive set theory [1, Remark 5.1]. We answer both questions in ZFC.

Theorem 1.1. *Let $1 \leq \alpha < \omega_1$. If $A \subseteq 2^\omega$ is absolutely Δ^1_2 and $A \in \Sigma^0_\alpha$, then A has a $\Sigma^{(<\omega_1)}_\alpha$-code in L. Likewise, if $A \in \Pi^0_\alpha$, then A has a $\Pi^{(<\omega_1)}_\alpha$-code in L.*

Corollary 1.2. *Every absolutely Δ^1_2 Borel set has a $\mathrm{Borel}^{(<\omega_1)}$-code in L.*

The obstacle to deriving the theorem directly from Stern's separation result is that, at Borel level α, the constructible code supplied there is indexed by $\omega^L_{\alpha^*}$, where $\alpha^* = \alpha - 1$ for finite α and $\alpha^* = \alpha$ for infinite α. This ordinal need not be countable in the ambient universe. We force it to become countable and apply Stern's theorem in the resulting extension. Absolute Δ^1_2-ness ensures that a fixed ordinary Borel code for A still separates the same two projectively defined sets there, so the theorem supplies a constructible generalized code of the required Borel class indexed by a now-countable ordinal. The point of the next section is that, for a fixed ordinary Borel code b, the existence of some equivalent constructible generalized code indexed by a countable ordinal and of a prescribed Borel class is a Σ^1_2 assertion about b. Shoenfield absoluteness then reflects this existence statement back to the ground model.

2. A Σ^1_2 reflection lemma for Borel codes

Fix a standard coding of ordinary Borel sets by reals. We code countable ordinals by reals coding well-orders on subsets of ω, so that finite ordinals are included. We also fix the canonical ordinal pairing function used in [1, Section 2.2]. Ordinal arithmetic and this pairing function are absolute between transitive models containing the relevant ordinals; in particular, multiplicatively closed ordinals are absolute.

We use the following standard facts: the set of ordinary Borel codes is Π^1_1; for ordinary Borel codes b, d, the assertion $B_b = B_d$ is Π^1_1; and, after an enumeration of a countable indexing ordinal is given, a generalized code can be converted by a uniform syntactic recursion into an ordinary code of the same Borel class. The first two facts are proved, for example, in [2, Lemmas 25.44 and 25.45]. The third is the recoding used in the reverse implication of [1, Lemma 5.5]. Consequently, once an enumeration of the indexing ordinal is fixed, membership in the set coded by a valid generalized code is uniformly Δ^1_1 in the resulting real presentation.

Lemma 2.1. *Let u code the countable ordinal $\alpha \geq 1$, and let b be an ordinary Σ^0_α-code. The following assertion is $\Sigma^1_2(b, u)$:*

> *There is a multiplicatively closed ordinal λ with $\omega \leq \lambda < \omega_1$ and a $\Sigma^{(\lambda)}_\alpha$-code $c \in L$ such that c codes B_b.*

The analogous assertion for Π^0_α-codes is also $\Sigma^1_2(b, u)$.

Proof. We first make the uniform recoding operation precise enough for the complexity calculation. Given a countable structure $M = (\omega, E)$, elements $\bar{c}, \bar{\lambda} \in M$, and a bijection e from ω onto the E-predecessors of $\bar{\lambda}$, let

$$\mathrm{Rec}(E, \bar{c}, \bar{\lambda}, e, d)$$

assert that d is obtained by transporting the labeled immediate-subcode tree determined by $\bar{c}$ along e. Thus the fixed constructor and basic-open clauses are preserved, while at every generalized union node the nth child of the corresponding ordinary node is obtained from the $e(n)$th generalized child. This transports the full syntax tree, rather than merely pulling back the underlying set $\bar{c}$, since e need not preserve the fixed pairing functions. Once the coding conventions are fixed, Rec is arithmetical in the displayed real parameters: every coordinate of d is determined by following a finite address through the relevant section operations. Moreover, the translation is an isomorphism of labeled immediate-subcode trees and therefore preserves validity, Borel class, and interpretation whenever the codes are externally well founded. This is the reindexing used in the reverse implication of [1, Lemma 5.5].

We now give a $\Sigma^1_2(b,u)$ formulation of the assertion, which is equivalent to saying that there exists a real coding the following:

(1) a countable well-founded extensional structure $M=(\omega,E)$, together with its full external satisfaction relation, such that the Tarski clauses hold and
$$M \models \mathrm{ZFC}^- + V = L,$$
where ZFC^- denotes ZFC without the Power Set axiom;

(2) elements $\bar{\alpha}, \bar{\lambda}, \bar{c} \in M$ such that M regards $\bar{\alpha}$ and $\bar{\lambda}$ as ordinals, $\bar{\lambda}$ as multiplicatively closed and at least ω, and $\bar{c}$ as a $\Sigma^{(\bar{\lambda})}_{\bar{\alpha}}$-code;

(3) an isomorphism from the well-order coded by u onto the E-predecessors of $\bar{\alpha}$, and a bijection e from ω onto the E-predecessors of $\bar{\lambda}$;

(4) a real d such that $\mathrm{Rec}(E,\bar{c},\bar{\lambda},e,d)$ holds and d is an externally valid ordinary Borel code.

And in addition that $B_d = B_b$.

Once the existential witnesses are fixed, the Tarski clauses, the assertion that every axiom in a fixed recursive presentation of $\mathrm{ZFC}^- + V = L$ holds, and all structural, isomorphism, enumeration, and recoding conditions are arithmetical in the displayed real parameters and in b, u. Well-foundedness of E, validity of d as an ordinary Borel code, and the equality $B_d = B_b$ are Π^1_1. The assertion therefore has the form
$$\exists z\, \forall y\ \vartheta(z,y,b,u),$$
where ϑ is arithmetical. Hence it is $\Sigma^1_2(b,u)$.

It remains to verify that this coded assertion is equivalent to the one in the lemma. Suppose first that the coded assertion holds. The transitive collapse of M is an initial segment of L. Let α', λ', and c' be the collapse images of $\bar{\alpha}$, $\bar{\lambda}$, and $\bar{c}$. Clause (3) gives $\alpha' = \alpha$, while $\lambda' < \omega_1$ because the collapse is countable. Multiplicative closure is absolute. By clause (4), the labeled immediate-subcode tree of c' is isomorphic to that of the externally valid ordinary code d and is therefore well founded. Since M regards $\bar{c}$ as a $\Sigma^{(\bar{\lambda})}_{\bar{\alpha}}$-code, induction on this well-founded tree shows that c' is an ambient $\Sigma^{(\lambda')}_{\alpha}$-code and that its ordinary recoding is d. Consequently,
$$c' \in L \qquad \text{and} \qquad B_{c'} = B_d = B_b.$$

Conversely, suppose that $c \in L$ is an ambient $\Sigma^{(\lambda)}_{\alpha}$-code for B_b, where $\omega \le \lambda < \omega_1$. Since the syntax of c is well founded in V, it is well founded in L, and the local constructor and Borel-level conditions are absolute. Choose a sufficiently large ordinal θ that is an uncountable regular cardinal in L such that
$$c, \lambda, \alpha \in L_\theta, \qquad L_\theta \models \mathrm{ZFC}^- + V = L,$$

and L_θ regards c as a $\Sigma^{(\lambda)}_\alpha$-code. Under the coding convention above, $c \subseteq \lambda$, so the transitive closure of $\{c, \lambda, \alpha\}$ is countable in the ambient universe. Choose a countable elementary substructure

$$X \prec L_\theta$$

that contains this transitive closure pointwise. Its Mostowski collapse therefore fixes c, λ, and α.

Let $e : \omega \to \lambda$ be a bijection, and let d be the ordinary code obtained by transporting the syntax of c along e. The uniform recoding preserves validity, Borel class, and interpretation, so d is an ordinary Σ^0_α-code and

$$B_d = B_c = B_b.$$

A real coding $(X, \in)$, its external satisfaction relation, the bijection e, and the code d supplies all the required witnesses.

The proof for Π-codes is identical. $\square$

3. Proof of the main theorem

For $1 \leq \alpha < \omega_1$, put

$$\alpha^* = \begin{cases} \alpha - 1, & \text{if } \alpha \text{ is finite,} \\ \alpha, & \text{if } \alpha \text{ is infinite.} \end{cases}$$

In the proof below, we use the following form of Stern's separation theorem [5, Section 0.2, Theorem 2]; see also [1, Theorem 5.6].

Theorem 3.1 (Stern)**.** *Let $1 \leq \alpha < \omega_1$. If two disjoint Σ^1_2 subsets of 2^ω are separated by a Σ^0_α set, then they are separated by a set having a $\Sigma^{(\omega^L_{\alpha^*})}_\alpha$-code in L.*

Proof of Theorem 1.1. Suppose that $A \in \Sigma^0_\alpha$. Fix an ordinary Σ^0_α-code b for A, and fix a real u coding a well-order of order type α. Let $\varphi(x)$ be a Σ^1_2 formula and $\psi(x)$ a Π^1_2 formula witnessing that A is absolutely Δ^1_2.

We first note that the same Borel code b continues to represent the set defined by φ and ψ after forcing. In the ground model,

$$\forall x\big(x \in B_b \longrightarrow \psi(x)\big), \tag{1}$$

$$\forall x\big(\varphi(x) \longrightarrow x \in B_b\big). \tag{2}$$

Each statement is $\Pi^1_2(b)$. For each inclusion, its failure asserts the existence of a real satisfying a $\Sigma^1_2(b)$ condition, since membership in the Borel set coded by b is $\Delta^1_1(b)$. By Shoenfield absoluteness [2, Theorem 25.20], both (1) and (2) persist to every set-forcing extension. Since the two definitions agree in every such extension, each extension W satisfies

$$B^W_b = \{x \in 2^\omega : W \models \varphi(x)\} = \{x \in 2^\omega : W \models \psi(x)\}. \tag{3}$$

Set

$$\kappa = \omega^L_{\alpha^*}.$$

Since κ is an infinite cardinal in L, it is multiplicatively closed; this remains true in all outer models by absoluteness of ordinal arithmetic. Force over V with $\mathrm{Col}(\omega, \kappa)$, and write $W = V[G]$. By (3), the two Σ^1_2 sets

$$A_0 = \{x \in 2^\omega : W \models \varphi(x)\} \qquad \text{and} \qquad A_1 = \{x \in 2^\omega : W \models \neg\psi(x)\}$$

are complementary, and the Σ^0_α set B^W_b separates them.

Apply Theorem 3.1 inside W. It gives a $\Sigma_\alpha^{(\omega^L_{\alpha^*})}$-code $c \in L$ for a separator of A_0 and A_1. Since these sets are complementary, the separator is exactly $A_0 = B_b^W$. By the displayed equalities, $c \in L$ and its indexing ordinal is κ, which is countable in W. Consequently, W satisfies the $\Sigma^1_2(b, u)$ assertion of Lemma 2.1. By Shoenfield absoluteness, that assertion already holds in V. The witnesses reflected to V need not be κ and c; what is obtained is some multiplicatively closed λ with $\omega \le \lambda < \omega_1^V$ and a corresponding code in L. Hence $A = B_b$ has a $\Sigma_\alpha^{(<\omega_1)}$-code in L.

If $A \in \Pi^0_\alpha$, then $2^\omega \setminus A$ is absolutely Δ^1_2 and belongs to Σ^0_α. Apply the preceding case and then the canonical syntactic complementation operation. This gives in L a $\Pi_\alpha^{(<\omega_1)}$-code for A with the same indexing ordinal. □

Proof of Corollary 1.2. Every Borel set belongs to Σ^0_α for some countable $\alpha \ge 1$, so the corollary follows from Theorem 1.1. □

Remark 3.2. The argument relativizes to a real parameter. If A is absolutely $\Delta^1_2(a)$ and $A \in \Sigma^0_\alpha$, then A has a $\Sigma_\alpha^{(<\omega_1)}$-code in $L[a]$. In Lemma 2.1, one uses countable well-founded models of $V = L[a]$; the relativized form of Stern's theorem supplies a code indexed by $\omega^{L[a]}_{\alpha^*}$.

Remark 3.3. For a fixed α, the proof only uses that the chosen Σ^1_2 and Π^1_2 definitions agree in the extension $V[G]$ used above. Agreement in all forcing extensions is therefore stronger than is needed at a fixed Borel level.

References

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