Countable OD sets in a Cohen extension contain only OD members: A proof by GPT-6 Astra

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Abstract

If c is Cohen-generic over L, then every countable ordinal-definable set in $L[c]$ has an ordinal-definable bijective enumeration. The same conclusion holds relative to any parameter that is a set of ordinals in $L[c]$. There is no restriction on the ranks of the members. The proof analyzes the fixed subalgebra of an automorphism action on a countable Boolean product, using the Kuratowski–Ulam theorem and the generic orbit-equivalence theorem of Sullivan–Weiss–Wright. It is entirely generated by GPT-6 Astra.

1 Introduction

Kanovei and Lyubetsky [5] proved that, in an extension by a single Cohen, random, dominating, or Sacks real, every countable ordinal-definable set of reals belongs to the ground model. Their Problem 8.1 asks whether, in these extensions, every nonempty countable ordinal-definable family of sets of reals contains an ordinal-definable member. The Sacks-over-L case has a negative answer. Enayat and Kanovei [1], revisiting an unpublished theorem of Solovay, constructed in such an extension an OD unordered pair of non-OD sets whose union is the set of nonconstructible reals. They obtained the same conclusion for an E_0-large generic extension of L.

We prove the following result for Cohen forcing. As usual, countable includes finite, and $\mathrm{OD}(a)$ denotes definability from a and ordinal parameters.

Theorem 1.1. *Let c be Cohen-generic over L. In $L[c]$, for every set of ordinals a, every countable $\mathrm{OD}(a)$ set has an $\mathrm{OD}(a)$ bijective enumeration by a finite ordinal or by ω. In particular, every member of a countable $\mathrm{OD}(a)$ set is $\mathrm{OD}(a)$.*

Taking $a = \varnothing$, every countable OD set in $L[c]$ has an OD enumeration. Note that there is no restriction on the ranks of the elements. Hence the specific case where the countable OD set is a family of sets of reals answers the Cohen-over-L case of [5, Problem 8.1], also posed in [4].

The forcing statement underlying Theorem 1.1 applies to an arbitrary ground model. Let M be a ground model of ZFC, let $\mathbb{B} = \mathrm{RO}(2^\omega)^M$ be its Cohen algebra, and let $\mathscr{G} = \mathrm{Aut}^M(\mathbb{B})$. A $\mathbb{B}$-name $\tau \in M$ is *invariant* if $1_{\mathbb{B}} \Vdash g\tau = \tau$ for every $g \in \mathscr{G}$. Thus invariance is equality in the Boolean-valued model, rather than literal equality of names.

Theorem 1.2. *Suppose that $\dot{\mathscr{A}} \in M$ is an invariant $\mathbb{B}$-name forced to be nonempty and countable. There are $I \in (\omega \setminus \{0\}) \cup \{\omega\}$ and a family $(\sigma_i)_{i\in I} \in M$ of invariant names such that the top condition forces $i \mapsto \sigma_i$ to be a bijection from $\check{I}$ onto $\dot{\mathscr{A}}$. The name for this enumeration is invariant as well.*

To prove Theorem 1.2, choose a name for a bijective enumeration $(\tau_i)_{i\in I}$ of $\dot{\mathscr{A}}$. The Boolean product $\mathbb{C} = \mathbb{B}^I$ represents forcing with $\mathbb{B}$ and then choosing a member of the enumerated family. The automorphism action on $\mathbb{B}$ lifts to $\mathbb{C}$. Its fixed algebra $\mathbb{D}$ is atomic, and the coordinates of each atom form a partition of $1_{\mathbb{B}}$. The resulting matrix of atoms gives an isomorphism that carries the lifted action to the coordinatewise action on $\mathbb{B}^I$. Mixing the names τ_i along the coordinates of an atom then gives an invariant name.

We prove this Boolean-algebra statement in Section 3, under an explicit locality assumption on the action. Atomicity follows from Kuratowski–Ulam. The disjointness of the coordinates uses Sullivan–Weiss–Wright, applied to Stone spaces of countable invariant subalgebras. Section 4 verifies locality for the action induced by names. The passage from invariant names to OD objects, carried out in Section 5, uses the definable well-order of $L[a]$; this last conclusion is not asserted for an arbitrary ground model.

2 Boolean algebras and category

All homomorphisms between complete Boolean algebras are assumed to preserve arbitrary joins. Indexed partitions may contain zero terms. For a Boolean algebra R, write $R^+ = R \setminus \{0\}$ and $R \restriction u = \{r \in R : r \le u\}$, with unit u. A subset of R^+ is *order-dense* if every nonzero element of R lies above one of its members. We use the usual Boolean-valued formulation of forcing; see [3].

An atomless complete Boolean algebra with a countable order-dense subset is isomorphic to the Cohen algebra. Indeed, a countable order-dense Boolean subalgebra is atomless, and the countable atomless Boolean algebra is unique up to isomorphism. We shall also use the identification of $\mathrm{RO}(X)$ with the Boolean algebra of Borel subsets of a Polish space X modulo meager sets. We write $[A]$ for the class of a Borel set A.

Lemma 2.1. *Suppose that a group H acts on a complete Boolean algebra R with a countable order-dense subset. The fixed algebra R^H is a complete subalgebra with a countable order-dense subset. There is a countable subgroup $H_0 \le H$ with $R^{H_0} = R^H$.*

Proof. Fixed elements are closed under complements and arbitrary joins. Let $(u_n)_{n<\omega}$ be order-dense in R^+, and put $s_n = \bigvee_{h\in H} h(u_n)$. Each s_n is fixed. If $0 < d \in R^H$ and $u_n \le d$, then $s_n \le d$, so the sequence (s_n) is order-dense in R^H.

The algebra R is ccc. Consequently each s_n is the join of $\{h(u_n) : h \in H_n\}$ for some countable $H_n \subseteq H$. Let H_0 be generated by $\bigcup_n H_n$. If $e \in R^{H_0}$ and $u_n \le e$, then $s_n \le e$. It follows that $e = \bigvee\{s_n : u_n \le e\} \in R^H$. The reverse inclusion is immediate. □

For $x, a \in 2^\omega$, let $x \oplus a$ denote coordinatewise addition modulo two.

Lemma 2.2. *Let I be nonempty and countable, let $r_i : 2^\omega \to 2^\omega$ be Borel for $i \in I$, and put $S(x) = \{r_i(x) : i \in I\}$. Suppose that $\{x : S(x) = S(x \oplus a)\}$ is comeager for every $a \in 2^\omega$. Then there is a countable set $S_* \subseteq 2^\omega$ such that $\{x : S(x) = S_*\}$ is comeager.*

Proof. The relation $E(x, y)$ defined by $S(x) = S(y)$ is Borel: it asserts that each $r_i(x)$ equals some $r_j(y)$, and conversely. The Borel set $\{(a, x) : E(x, x \oplus a)\}$ has comeager vertical sections and hence is comeager by Kuratowski–Ulam [7, Theorem 8.41]. The map $(a, x) \mapsto (x, x \oplus a)$ is a homeomorphism of $(2^\omega)^2$. Therefore E is comeager. A second application of Kuratowski–Ulam gives x_* for which E_{x_*} is comeager; take $S_* = S(x_*)$. □

We need one consequence of the theorem of Sullivan–Weiss–Wright. For $H \leq \mathrm{Aut}(R)$, say that an isomorphism $\varphi : R \upharpoonright u \to R \upharpoonright v$ is *piecewise given by* H if there are a countable partition (u_n) of u and elements $h_n \in H$ such that $\varphi(z) = h_n(z)$ whenever $z \leq u_n$. The elements $h_n(u_n)$ then partition v. Inverses and compositions of such isomorphisms have the same property: use the image partition for an inverse, and refine the domain partition by inverse images of the second partition for a composition.

Lemma 2.3. *Let R and S be atomless complete Boolean algebras with countable order-dense subsets. Suppose that $H \leq \mathrm{Aut}(R)$ and $K \leq \mathrm{Aut}(S)$ are countable, with $R^H = \{0,1\}$ and $S^K = \{0,1\}$. There is a complete isomorphism $\alpha : R \to S$ such that $\alpha h \alpha^{-1}$ is piecewise given by K for every $h \in H$.*

Proof. Choose countable order-dense Boolean subalgebras $R_0 \subseteq R$ and $S_0 \subseteq S$, invariant under H and K, respectively. Their Stone spaces X and Y are Cantor spaces, and the induced actions are by homeomorphisms. Identifying their regular open algebras with R and S, every invariant Borel set has class 0 or 1. Thus both actions are generically ergodic.

By Sullivan–Weiss–Wright [9, Theorem 1.8], the two actions are orbit equivalent through a homeomorphism $f : X_0 \to Y_0$ between invariant dense G_δ subspaces. The two-space form follows by identifying each action with the canonical finite-bit-flip action on a dense G_δ subset of Cantor space and intersecting those two invariant subsets. Direct image under f induces an isomorphism $\alpha : R \to S$.

Fix $h \in H$, and enumerate K as (k_n). Partition Y_0 into the Borel sets on which $fhf^{-1}(y) = k_n(y)$ for the least such n. Their classes, after zero classes are discarded, partition 1_S. On the ideal below each of these classes, $\alpha h \alpha^{-1}$ agrees with the corresponding k_n. This gives the conclusion. □

3 Actions on countable Boolean products

In this section $\mathbb{B} = \mathrm{RO}(2^\omega)$, $\mathscr{G} = \mathrm{Aut}(\mathbb{B})$, and I is a nonempty countable set. Let $\mathbb{C} = \mathbb{B}^I$ be the Boolean direct product, with diagonal embedding $\delta : \mathbb{B} \to \mathbb{C}$ and slice units e_i, where $(e_i)_i = 1$ and $(e_i)_j = 0$ for $j \neq i$. The algebra $\mathbb{C}$ is atomless and has a countable order-dense subset, obtained by placing a basic Cohen condition in a single coordinate.

For $z \in \mathbb{C}$, put $s(z) = \bigvee_{i \in I} z_i$, and define $\iota_z : \mathbb{B} \upharpoonright s(z) \to \mathbb{C} \upharpoonright z$ by $\iota_z(b) = \delta(b) \wedge z$. This is a complete embedding. It is onto exactly when the coordinates of z are pairwise disjoint. In that case its inverse is $w \mapsto s(w)$. Conversely, if it is onto, write $z \wedge e_i = \iota_z(b_i)$. Then $z_i \leq b_i$ and $b_i \wedge z_j = 0$ for $j \neq i$, which proves the disjointness.

Suppose that $\Theta : \mathscr{G} \to \mathrm{Aut}(\mathbb{C})$ is a homomorphism satisfying

$$\Theta_g(\delta(b)) = \delta(g(b)) \qquad (g \in \mathscr{G},\ b \in \mathbb{B}) \tag{3.1}$$

and the following locality condition:

$$g \upharpoonright (\mathbb{B} \upharpoonright b) = \mathrm{id} \quad \Longrightarrow \quad \Theta_g \upharpoonright (\mathbb{C} \upharpoonright \delta(b)) = \mathrm{id}\,. \tag{3.2}$$

If g and h agree on $\mathbb{B} \upharpoonright b$, applying (3.2) to $h^{-1}g$ shows that Θ_g and Θ_h agree on $\mathbb{C} \upharpoonright \delta(b)$.

The least diagonal element above z is $\delta(s(z))$. Hence (3.1) gives $s(\Theta_g z) = g(s(z))$. It also gives $\Theta_g \iota_z(b) = \iota_{\Theta_g z}(g(b))$, so Θ_g preserves the property that the coordinates of an element are pairwise disjoint.

Put $\mathbb{D} = \mathbb{C}^{\Theta[\mathscr{G}]}$. Every nonzero $d \in \mathbb{D}$ satisfies $s(d) = 1$. Indeed, $s(d)$ is fixed by every automorphism of $\mathbb{B}$, and $\mathbb{B}^{\mathscr{G}} = \{0, 1\}$. For the latter assertion, if $0 < b < 1$, choose basic cylinders of the same length below b and $\neg b$. A finite translation carrying the first cylinder to the second does not fix b.

Theorem 3.1. *The algebra $\mathbb{D}$ is atomic, and its atoms can be enumerated as $(d_k)_{k\in I}$ so that every row and every column of the matrix $(d_{k,i})_{k,i\in I}$ is a partition of $1_{\mathbb{B}}$. The map*

$$W : \mathbb{B}^I \longrightarrow \mathbb{C}, \qquad W((b_k)_{k\in I}) = \bigvee_{k\in I} (\delta(b_k) \wedge d_k) \tag{3.3}$$

is a complete Boolean isomorphism fixing the diagonal. If g^I denotes coordinatewise application of g, then $\Theta_g W = W g^I$ for every $g \in \mathscr{G}$.

We first prove atomicity. Only (3.1), not (3.2), is used in this part.

Proposition 3.2. *The algebra $\mathbb{D}$ is atomic.*

Proof. By Lemma 2.1, choose an order-dense sequence $(v_n)_{n<\omega}$ in $\mathbb{D}^+$. Represent its coordinates by regular open sets $V_{n,i}$, and define Borel maps $r_i : 2^\omega \to 2^\omega$ by $r_i(x)(n) = 1$ exactly when $x \in V_{n,i}$. Put $S(x) = \{r_i(x) : i \in I\}$.

Fix $a \in 2^\omega$, and let $t_a \in \mathscr{G}$ be induced by $x \mapsto x \oplus a$. Write $m^a_{ij} = (\Theta_{t_a} e_i)_j$. Each row of this matrix partitions $1_{\mathbb{B}}$, because e_i has pairwise disjoint coordinates and support 1, and both properties are preserved by Θ_{t_a}. Each column partitions $1_{\mathbb{B}}$ because $(e_i)_{i\in I}$ partitions $1_{\mathbb{C}}$. Since v_n is fixed and $v_n = \bigvee_i (e_i \wedge \delta(v_{n,i}))$,

$$m^a_{ij} \wedge t_a(v_{n,i}) = m^a_{ij} \wedge v_{n,j} \qquad (n < \omega,\ i, j \in I). \tag{3.4}$$

In the category algebra, taking all n in (3.4) gives $m^a_{ij} \le [\{x : r_i(x \oplus a) = r_j(x)\}]$. The row and column partitions therefore imply that $S(x \oplus a) = S(x)$ on a comeager set. By Lemma 2.2, there is a countable $S_* \subseteq 2^\omega$ for which $S(x) = S_*$ on a comeager set.

For $t \in S_*$, define

$$a_t = \bigwedge_{n<\omega} v_n^{t(n)}, \qquad v_n^1 = v_n, \quad v_n^0 = \neg_{\mathbb{C}} v_n. \tag{3.5}$$

These elements belong to $\mathbb{D}$, and $(a_t)_i = [\{x : r_i(x) = t\}]$. Consequently $\bigvee_{t\in S_*} a_t = 1_{\mathbb{C}}$. Distinct a_t's are disjoint. If $0 < z \le a_t$ with $z \in \mathbb{D}$, choose n such that $0 < v_n \le z$. Since a_t lies below either v_n or its complement, we must have $a_t \le v_n$. Thus $a_t \le v_n \le z \le a_t$, and $z = a_t$. Every nonzero a_t is therefore an atom of $\mathbb{D}$. □

The next lemma compares an arbitrary isomorphism with the diagonal embedding. All instances of "piecewise" refer to the definition in Section 2, applied to subgroups of $\mathrm{Aut}(\mathbb{C})$.

Lemma 3.3. *Let $0 < d \in \mathbb{D}$, and let $\alpha : \mathbb{B} \to \mathbb{C} \restriction d$ be a complete isomorphism. There exist $q \le d$, with coordinates partitioning $1_{\mathbb{B}}$, and an isomorphism $U : \mathbb{C} \restriction q \to \mathbb{C} \restriction d$, piecewise given by $\Theta[\mathscr{G}]$, such that $\alpha = U\iota_q$.*

Proof. The nonzero elements among $\alpha^{-1}(d \wedge e_i)$ partition $1_{\mathbb{B}}$. Split each into two nonzero pieces and enumerate the resulting partition as (p_n). Let i_n be the slice containing $\alpha(p_n)$. The map $b \mapsto \alpha(b)_{i_n}$ is an isomorphism from $\mathbb{B} \restriction p_n$ onto $\mathbb{B} \restriction \alpha(p_n)_{i_n}$. Both units are proper and nonzero.

Their complementary ideals are Cohen algebras, so this isomorphism extends to some $g_n \in \mathscr{G}$. Thus

$$\alpha(b) = e_{i_n} \wedge \delta(g_n(b)) \qquad (b \leq p_n). \tag{3.6}$$

Put $u_n = \Theta_{g_n^{-1}}(\alpha(p_n))$ and $q = \bigvee_n u_n$. Since $\alpha(p_n)$ lies in one slice, its coordinates are pairwise disjoint; the same holds for u_n. Moreover, $s(u_n) = g_n^{-1}(\alpha(p_n)_{i_n}) = p_n$. The supports p_n are disjoint and have join 1, so the coordinates of q partition $1_{\mathbb{B}}$. Each u_n lies below d, since $\alpha(p_n) \leq d$ and d is fixed by $\Theta_{g_n^{-1}}$.

The maps Θ_{g_n} carry the partition (u_n) of q to the partition $(\alpha(p_n))$ of d. Hence $U(z) = \bigvee_n \Theta_{g_n}(z \wedge u_n)$ defines an isomorphism $\mathbb{C}\restriction q \to \mathbb{C}\restriction d$ piecewise given by $\Theta[\mathscr{G}]$. For $b \leq p_n$, we have $\iota_q(b) = \delta(b) \wedge u_n$, and (3.1) and (3.6) give $U\iota_q(b) = \alpha(b)$. Taking joins over n proves $U\iota_q = \alpha$. □

Proof of Theorem 3.1. Choose, by Lemma 2.1, a countable subgroup $\Gamma \leq \mathscr{G}$ such that $\mathbb{C}^{\Theta[\Gamma]} = \mathbb{D}$. If $b \in \mathbb{B}$ is fixed by Γ, then $\delta(b) \in \mathbb{D}$, so b is fixed by $\mathscr{G}$. Thus $\mathbb{B}^{\Gamma} = \{0, 1\}$.

Fix an atom $d \in \mathbb{D}$, whose existence follows from Proposition 3.2. The restricted action of $\Theta[\Gamma]$ on $\mathbb{C}\restriction d$ has fixed algebra $\{0, d\}$. Apply Lemma 2.3 to obtain an isomorphism $\alpha : \mathbb{B} \to \mathbb{C}\restriction d$ such that, for each $\gamma \in \Gamma$, the automorphism $\beta_\gamma = \alpha\gamma\alpha^{-1}$ is piecewise given by the restrictions of members of $\Theta[\Gamma]$.

By Lemma 3.3, write $\alpha = U\iota_q$, where $q \leq d$ has coordinates partitioning $1_{\mathbb{B}}$ and U is piecewise given by $\Theta[\mathscr{G}]$. For $\gamma \in \Gamma$, the automorphism

$$V_\gamma = U^{-1}\beta_\gamma U = \iota_q \gamma \iota_q^{-1} \tag{3.7}$$

of $\mathbb{C}\restriction q$ is again piecewise given by $\Theta[\mathscr{G}]$.

Let $u \leq q$ be a piece on which V_γ agrees with Θ_g. Write $u = \iota_q(b)$. If $a \leq b$, then (3.7) yields $\Theta_g\iota_q(a) = \iota_q(\gamma(a))$. Taking supports gives $g(a) = \gamma(a)$. By locality, Θ_g and Θ_γ therefore agree on $\mathbb{C}\restriction \delta(b)$, and in particular on $\mathbb{C}\restriction u$. Applying this to every piece shows that $V_\gamma(z) = \Theta_\gamma(z)$ for all $z \leq q$. Hence $\Theta_\gamma(q) = V_\gamma(q) = q$.

It follows that $q \in \mathbb{C}^{\Theta[\Gamma]} = \mathbb{D}$. Since $0 < q \leq d$ and d is an atom, $q = d$. Thus the coordinates of every atom of $\mathbb{D}$ partition $1_{\mathbb{B}}$.

Let $J = \mathrm{At}(\mathbb{D})$. This set is countable because $\mathbb{C}$ is ccc. By atomicity its members partition $1_{\mathbb{C}}$, so every column of the matrix $(d_i)_{d\in J,\ i\in I}$ also partitions $1_{\mathbb{B}}$. These row and column partitions imply $|J| = |I|$. To see the finite case, if there are n columns and $n+1$ rows, distribute the meet of the $n+1$ row joins. Every term contains two entries from the same column and is zero, a contradiction. Transposing gives the reverse inequality when J is finite. Since both index sets are countable, this proves the assertion.

Enumerate J by I. Each map $\iota_{d_k} : \mathbb{B} \to \mathbb{C}\restriction d_k$ is an isomorphism, and the d_k's partition $1_{\mathbb{C}}$. Thus the map W in (3.3) is an isomorphism, with inverse $z \mapsto (s(z \wedge d_k))_{k\in I}$. It fixes the diagonal. Finally, Θ_g fixes each d_k and acts by g on diagonal elements, which gives $\Theta_g W = W g^I$. □

4 Invariant names

We now work inside a ground model M of ZFC. The algebras, groups, products, and joins in this section are computed in M. In particular, Theorem 3.1 is applied in M to its Cohen algebra $\mathbb{B} = \mathrm{RO}(2^\omega)^M$ and its full automorphism group $\mathscr{G} = \mathrm{Aut}^M(\mathbb{B})$.

Let $\dot{\mathscr{A}}$ be as in Theorem 1.2. Since it is invariant, the Boolean values asserting its possible finite or countably infinite cardinalities are fixed by $\mathscr{G}$ and hence belong to $\{0, 1\}$. Choose

$I \in (\omega \setminus \{0\}) \cup \{\omega\}$ and names $(\tau_i)_{i\in I}$ such that the top condition forces $(\tau_i)_{i\in I}$ to enumerate $\dot{\mathscr{A}}$ bijectively.

As before let $\mathbb{C} = \mathbb{B}^I$, with diagonal embedding δ and slice units e_i. This direct product is the completion of a lottery sum of copies of $\mathbb{B}$, rather than a product forcing adding independent generics. A generic filter on $\mathbb{C}$ meets exactly one slice unit e_i; its restriction to that slice is a generic filter G on $\mathbb{B}$. It is therefore $\{z \in \mathbb{C} : z_i \in G\}$, and conversely every such filter is generic. The selected member of $\dot{\mathscr{A}}$ is τ_i^G.

For $g \in \mathscr{G}$, put $m^g_{ij} = \|g\tau_i = \tau_j\|_{\mathbb{B}}$. Every row and column of (m^g_{ij}) partitions $1_{\mathbb{B}}$, because both $(g\tau_i)_{i\in I}$ and $(\tau_i)_{i\in I}$ are forced bijective enumerations of $\dot{\mathscr{A}}$. Define

$$(\Theta_g z)_j = \bigvee_{i\in I} (g(z_i) \wedge m^g_{ij}). \tag{4.1}$$

Lemma 4.1. *The map $g \mapsto \Theta_g$ is a homomorphism from $\mathscr{G}$ to* $\mathrm{Aut}(\mathbb{C})$ *satisfying* (3.1) *and* (3.2).

Proof. The row and column partitions show that each Θ_g preserves complements and arbitrary joins. Resolving equality through the intermediate enumeration gives

$$m^{gh}_{ik} = \bigvee_{j\in I} (g(m^h_{ij}) \wedge m^g_{jk}). \tag{4.2}$$

Consequently $\Theta_{gh} = \Theta_g\Theta_h$. Also $\Theta_1 = \mathrm{id}$, so $\Theta_{g^{-1}}$ is the inverse of Θ_g. The column partitions give (3.1).

Suppose that g is the identity on $\mathbb{B} \restriction b$. Induction on names gives $b \Vdash g\tau = \tau$ for every $\mathbb{B}$-name τ: for each coefficient p, one has $b \wedge g(p) = b \wedge p$, and the same induction applies to its associated name. It follows that $b \wedge m^g_{ii} = b$ and $b \wedge m^g_{ij} = 0$ for $i \neq j$. Formula (4.1) then shows that Θ_g is the identity below $\delta(b)$. □

For a name σ forced to belong to $\dot{\mathscr{A}}$, let $q_\sigma = (\|\sigma = \tau_i\|_{\mathbb{B}})_{i\in I}$. Its coordinates partition $1_{\mathbb{B}}$, and equality of two such elements implies forced equality of the corresponding names. The definition of the lift gives

$$\Theta_g(q_\sigma) = q_{g\sigma}. \tag{4.3}$$

Indeed, its jth coordinate is $\bigvee_i (\|g\sigma = g\tau_i\|_{\mathbb{B}} \wedge \|g\tau_i = \tau_j\|_{\mathbb{B}}) = \|g\sigma = \tau_j\|_{\mathbb{B}}$.

Proof of Theorem 1.2. Apply Theorem 3.1 to the action in Lemma 4.1, and enumerate the atoms of its fixed algebra as $(d_k)_{k\in I}$. For each k, use the mixing lemma to choose σ_k with $d_{k,i} \Vdash \sigma_k = \tau_i$ for every $i \in I$. The row partitions and the forced distinctness of the τ_i's give $q_{\sigma_k} = d_k$. Since d_k is fixed, (4.3) implies $1_{\mathbb{B}} \Vdash g\sigma_k = \sigma_k$ for every $g \in \mathscr{G}$.

For $k \neq \ell$, the Boolean value of $\sigma_k = \sigma_\ell$ is $\bigvee_i (d_{k,i} \wedge d_{\ell,i}) = 0$. For each i, the equality $\bigvee_k d_{k,i} = 1_{\mathbb{B}}$ forces τ_i to equal some σ_k. Thus the top condition forces the required bijection. Its name is invariant because each coordinate name is invariant and its index set is a ground-model set. □

When $\dot{\mathscr{A}}$ is forced to be a set of reals, its invariant enumeration can be coded by an invariant name for a real. Every Boolean value deciding a bit of this real is fixed by $\mathscr{G}$, and hence is 0 or 1. The code, and therefore the enumerated set, belongs to M. In particular, Theorem 1.2 recovers the Cohen case of [5, Theorem 1.1] over an arbitrary ground model: an OD definition uses ordinal parameters fixed by every ground-model automorphism, and homogeneity gives an invariant name for the set.

5 Ordinal definability and parameters

We use the same-extension theorem of Vopěnka–Hájek and Grigorieff: if R is a complete Boolean algebra in M and G, H are M-generic filters with $M[G] = M[H]$, then some $g \in \mathrm{Aut}^M(R)$ carries G to H; see [2, Theorem 3.5.1] and [8, Theorem 2.16]. The action on names is covariant: $(g\tau)^{gG} = \tau^G$.

Lemma 5.1. *Let $M = L[a]$, where a is a set of ordinals, and let $R \in M$ be a complete Boolean algebra. If G is M-generic and $\tau \in M$ is an R-name invariant under $\mathrm{Aut}^M(R)$, then τ^G is $\mathrm{OD}(a)$ in $M[G]$.*

Proof. Work in $V = M[G]$. Every member of M, including R and τ, is $\mathrm{OD}(a)$ using its ordinal position in the canonical well-order of $L[a]$. Let T consist of all values τ^H where $H \subseteq R$ is M-generic and $M[H] = V$. This defines a set from a, R, τ. The last condition is first-order: every set is the value under H of an R-name belonging to the definable class M. Thus T is $\mathrm{OD}(a)$. For every such H, the same-extension theorem gives $g \in \mathrm{Aut}^M(R)$ with $H = gG$. Invariance and covariance give $\tau^H = (g\tau)^{gG} = \tau^G$. Hence $T = \{\tau^G\}$, as required. □

Corollary 5.2. *If c is Cohen-generic over $L[a]$, where a is a set of ordinals, then every countable $\mathrm{OD}(a)$ set in $L[a, c]$ has an $\mathrm{OD}(a)$ bijective enumeration by a finite ordinal or by ω.*

Proof. Let A be nonempty and countable, uniquely defined in $L[a, c]$ by a formula $\varphi(A, a, \vec{\xi})$ with ordinal parameters $\vec{\xi}$. The Boolean value of the assertion that φ defines a unique nonempty countable set is fixed by all automorphisms of the Cohen algebra in $L[a]$. It is nonzero, since the assertion holds in the given extension, and therefore equals 1. Choose a name $\dot{\mathscr{A}}$ forced to satisfy that definition. Every automorphism fixes $\check{a}$ and the ordinal parameters; uniqueness consequently gives $1 \Vdash g\dot{\mathscr{A}} = \dot{\mathscr{A}}$. Apply Theorem 1.2 in $L[a]$ and Lemma 5.1 to the invariant enumeration name. The empty set has the empty enumeration. □

To apply Corollary 5.2 to an arbitrary set of ordinals $a \in L[c]$, we use the following folklore fact about intermediate extensions. See [6].

Lemma 5.3. *Suppose that c is Cohen-generic over L and that $a \in L[c]$ is a set of ordinals. Either $L[a] = L[c]$, or $L[c]$ is a Cohen-real extension of $L[a]$.*

Proof. By the intermediate model theorem [3, Chapter 15], there is a complete subalgebra B_0 of the Cohen algebra $\mathbb{B}$ in L such that $L[a] = L[H]$ for the induced generic filter H on B_0. Let $\pi : \mathbb{B} \to B_0$ be the projection taking b to the least member of B_0 above it, and fix a countable order-dense subset $P \subseteq \mathbb{B}^+$ in L. The quotient forcing over $L[H]$ can be represented by $Q = \{b \in \mathbb{B}^+ : \pi(b) \in H\}$ with the inherited order.

The set $P \cap Q$ is dense in Q. Indeed, if $b \in Q$, then $\pi(b) = \bigvee\{\pi(p) : p \in P,\ p \leq b\}$. Genericity of H for B_0, applied to this ground-model join, gives $p \leq b$ with $\pi(p) \in H$. Thus the completion of Q has a countable order-dense subset in $L[H]$. If the quotient generic contains an atom, it adds nothing. Otherwise it contains the complement of the join of all atoms; the algebra below that element is an atomless complete Boolean algebra with a countable order-dense subset, and hence is a Cohen algebra. This proves the alternatives. □

Proof of Theorem 1.1. Fix a set of ordinals $a \in L[c]$. If $L[a] = L[c]$, use the canonical well-order of $L[a]$ to choose the least bijective enumeration of the given family by a finite ordinal or by ω. Otherwise Lemma 5.3 makes $L[c]$ a Cohen extension of $L[a]$, and Corollary 5.2 applies. □

Theorem 1.1 also yields a uniformization consequence. The second coordinates in the next statement may again have arbitrary rank.

Corollary 5.4. *Work in $L[c]$. Let a be a set of ordinals, and let $R \in \mathrm{OD}(a)$ be a set of ordered pairs whose first coordinates are reals and whose vertical sections are countable. There is an $\mathrm{OD}(a)$ sequence $(f_n)_{n<\omega}$ of functions on $D = \mathrm{dom}(R)$ such that $R_x = \{f_n(x) : n < \omega\}$ for every $x \in D$. In particular, R has an $\mathrm{OD}(a)$ uniformization.*

Proof. For each $x \in D$, the nonempty set R_x is $\mathrm{OD}(a, x)$. The pair (a, x) can be uniformly coded by a set of ordinals. By Theorem 1.1, there is an $\mathrm{OD}(a, x)$ surjection $e_x : \omega \to R_x$, allowing repetitions when R_x is finite. Choose the least such surjection in the canonical definable well-order of $\mathrm{OD}(a, x)$. This choice is uniform in a, x and the ordinal parameters defining R. Replacement gives the $\mathrm{OD}(a)$ assignment $x \mapsto e_x$. Set $f_n(x) = e_x(n)$. The empty domain causes no exception. □

6 AI Use Disclaimer

This proof is entirely generated by GPT-6 Astra. The author, while waiting for his laundry, gave the model the following prompt:

> Let's try this problem again (link pasted). A while ago we attempted this and arrived at many constraints and obstructions, but no positive direction. I want to see how Astra would do differently. It's late and I'll be going to bed, so work for as long as you possibly can and do not stop unless you've found a solution or something presentable

Astra found the proof in 49m 52s. Laundry was done 10 minutes later. The proof was subsequently checked and tightened up by the author, who assumes all of the responsibility and none of the credit.

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