THE DESCRIPTIVE COMPLEXITY OF ERGODIC MEASURE-PRESERVING TRANSFORMATIONS THAT ADMIT ROOTS

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Abstract. Let $\mathrm{Aut}([0,1],\lambda)$ be the Polish group of invertible measure-preserving transformations with the weak topology. Using Kunde's reduction from ill-founded trees to reversibility of weakly mixing transformations, we apply a skew-product construction over an odometer to show that, for each integer $n \geq 2$, the set of ergodic transformations that admit an nth root is complete analytic. The same is therefore true without the ergodicity requirement. This answers a question by Thornton.

1. Introduction

Let (X,μ) be a standard atomless probability space, and let $G = \mathrm{Aut}(X,\mu)$ be the group of invertible measure-preserving transformations modulo equality almost everywhere, equipped with the weak topology. We write $R \cong Q$ if R and Q are measure-theoretically conjugate. For $n \geq 2$, set $\mathcal{R}_n = \{T \in G : (\exists S \in G)\ S^n = T\}$ and $\mathcal{R}_n^{\mathrm{erg}} = \{T \in \mathcal{R}_n : T \text{ is ergodic}\}$. The map $S \mapsto S^n$ is continuous, so $\mathcal{R}_n$ is analytic. The ergodic transformations form a G_δ subset of G, and hence $\mathcal{R}_n^{\mathrm{erg}}$ is analytic as well. Notice also that every root of an ergodic transformation is ergodic, since every set invariant under the root is invariant under its nth power.

The main result is the following.

Theorem 1.1. *For every integer $n \geq 2$, the set $\mathcal{R}_n^{\mathrm{erg}}$ is $\mathbf{\Sigma}_1^1$-complete.*

The reduction in the proof takes values in the ergodic transformations. It therefore gives the unrestricted statement at the same time.

Corollary 1.2. *For every integer $n \geq 2$, the set $\mathcal{R}_n$ is $\mathbf{\Sigma}_1^1$-complete.*

Roots of measure-preserving transformations were studied by Halmos [6]. Ornstein subsequently constructed a mixing transformation with no square root [11], whereas King proved that a generic transformation has roots of every order [8]. For $n = 2$, Corollary 1.2 proves the descriptive-set-theoretic conjecture in Problem 43 of [1].

Foreman, Rudolph, and Weiss proved both that reversibility is complete analytic among ergodic transformations and that the ergodic transformations with nontrivial centralizer form a complete analytic set [3]. However, there is no straightforward way to apply their result directly to Theorem 4.6

below to derive our result: the converse in that theorem is proved only for weakly mixing inputs, whereas the Foreman–Rudolph–Weiss theorem does not provide a reduction with weakly mixing range. We therefore use Kunde's refinement [9, Theorem 3], which gives a continuous family $\tau \mapsto R_\tau$ of weakly mixing transformations such that

$$\tau \text{ is ill-founded} \quad \Longleftrightarrow \quad R_\tau \cong R_\tau^{-1}.$$

For each fixed $n \geq 2$, we construct a continuous map $R \longmapsto \mathcal{G}_n(R)$ on the weakly mixing transformations such that $\mathcal{G}_n(R)$ is ergodic and

$$\mathcal{G}_n(R) \text{ has an } n\text{th root} \quad \Longleftrightarrow \quad R \cong R^{-1}.$$

Composing the two maps proves Theorem 1.1.

The map $\mathcal{G}_n(R)$ is a skew product over an odometer B on a profinite group Z, with fiber X^n. Its cocycle takes values in $\Gamma = \mathbb{Z}^n$; each of the first $n-1$ standard generators acts by R on the corresponding coordinate and trivially on the others, while the last acts by R^{-1} on the last coordinate. The cocycle is chosen so that, when R is weakly mixing, the extension is relatively weakly mixing; consequently the base is the maximal Kronecker factor. If $URU^{-1} = R^{-1}$, a cyclic permutation of the fiber coordinates, twisted by U, gives an explicit nth root. Conversely, any nth root descends to the unique nth root of B, and the resulting fiber equations force $R \cong R^{-1}$.

For background on measurable cocycles and skew-product extensions, see [12]. We use the standard characterization of relative weak mixing by the relatively independent square, as well as basic facts about Kronecker factors; see [5, Chapter 9] and, for the structural role of relative weak mixing, [4].

2. A cocycle over an odometer

Fix $n \geq 2$ for the remainder of Sections 2–4. Let $\Gamma = \mathbb{Z}^n$ with standard basis $e_0, \dots, e_{n-1}$, and let $P \in \mathrm{Aut}(\Gamma)$ be the cyclic permutation determined by $Pe_i = e_{i+1}$, with indices taken modulo n.

Put $c = 4n+1$ and $d_j = n(j+1)+1$ for $j \geq 0$, and define positive integers q_j and m_j by

$$q_0 = 1, \qquad m_j = cq_j, \qquad q_{j+1} = d_j m_j.$$

Then $m_j \mid q_{j+1} \mid m_{j+1}$, and every q_j and m_j is relatively prime to n. In particular, $m_r \mid q_j$ whenever $r < j$. Let $Z = \varprojlim_j \mathbb{Z}/m_j\mathbb{Z}$ with Haar probability measure ν, and let $\pi_j : Z \to \mathbb{Z}/m_j\mathbb{Z}$ be the coordinate maps. We use additive notation and write $\overline{r}$ for the image in Z of $r \in \mathbb{Z}$.

Define $A(z) = z + \overline{1}$ and $B(z) = z + \overline{n} = A^n(z)$. The image of $\mathbb{Z}$ is dense in Z. Multiplication by n is an automorphism of every quotient $\mathbb{Z}/m_j\mathbb{Z}$, and hence of Z. It follows that B is ergodic and that Z has no nonzero element annihilated by n.

For $j \geq 0$, let $D_j^+ = \pi_j^{-1}(\{0\})$ and $D_j^- = \pi_j^{-1}(\{[2nq_j]_{m_j}\})$. These sets are disjoint because $2nq_j < m_j$. Define $k_j = e_0\big(\mathbf{1}_{D_j^+} - \mathbf{1}_{D_j^-}\big)$. Since $m_j \geq c^{j+1}$,

$$\sum_{j=0}^{\infty} \nu(D_j^+ \cup D_j^-) = \sum_{j=0}^{\infty} \frac{2}{m_j} < \infty.$$

Thus the series $\sum_j k_j(z)$ is finite for almost every z. Let $N_\infty = \limsup_j (D_j^+ \cup D_j^-)$, so that $\nu(N_\infty) = 0$, and define

$$k(z) = \begin{cases} \displaystyle\sum_{j=0}^{\infty} k_j(z), & z \notin N_\infty, \\ 0, & z \in N_\infty. \end{cases}$$

Finally, define a measurable function $\phi : Z \to \Gamma$ by

$$\phi(z) = \sum_{a=0}^{n-1} P^{-a} k(A^a z). \tag{1}$$

The function ϕ generates a Γ-valued additive cocycle over B; for $q \geq 1$, write

$$\phi^{[q]}(z) = \sum_{r=0}^{q-1} \phi(B^r z). \tag{2}$$

All later uses of the explicit cocycle are concentrated in the following return property, which realizes every generator e_i on a return inside any set of positive measure.

Lemma 2.1. *The set $K = \{z \in Z : k(z) = 0\}$ has positive measure. Moreover, if $E \subseteq Z$ is measurable with $\nu(E) > 0$ and $i < n$, then there is $q \geq 1$ such that*

$$\nu\big(E \cap B^{-q}E \cap \{\phi^{[q]} = e_i\}\big) > 0.$$

Proof. The complement of $\bigcup_j (D_j^+ \cup D_j^-)$ is contained in K. Since

$$\sum_{j=0}^{\infty} \frac{2}{m_j} \leq \sum_{j=0}^{\infty} \frac{2}{c^{j+1}} = \frac{2}{c-1} < 1,$$

we have $\nu(K) > 0$.

Fix $i < n$, and let $a_i \in \{0, \ldots, n-1\}$ be determined by $P^{-a_i} e_0 = e_i$. We use the return times q_j. Since $B = A^n$, equations (1) and (2) give

$$\phi^{[q_j]}(z) = \sum_{t=0}^{nq_j - 1} P^{-t} k(A^t z). \tag{3}$$

Let

$$E_{j,i} = \bigcup_{s=0}^{q_j - 1} A^{-(a_i + ns)} D_j^+. \tag{4}$$

The sets in this union are distinct cosets of D_j^+, because multiplication by n is invertible modulo m_j and $q_j < m_j$. Therefore $\nu(E_{j,i}) = q_j/m_j = 1/c$. For $z \in E_{j,i}$, the orbit segment $z, Az, \ldots, A^{nq_j-1}z$ meets D_j^+ exactly once, at a time congruent to a_i modulo n. It does not meet D_j^-. Indeed, if it met both sets at times t_+ and t_-, then $t_- - t_+ \equiv 2nq_j \pmod{m_j}$. But $|t_- - t_+| < nq_j$, whereas the two representatives $2nq_j$ and $2nq_j - m_j = -(2n+1)q_j$ have absolute value greater than nq_j. Thus the contribution of k_j to (3) is e_i on $E_{j,i}$.

Every lower level contributes zero. Fix $\ell < j$. The sequence $t \mapsto P^{-t}k_\ell(A^t z)$ has period nm_ℓ, since n and m_ℓ are relatively prime. The length nq_j is a multiple of nm_ℓ. Over one period, the Chinese remainder theorem gives

$$\sum_{t=0}^{nm_\ell-1} P^{-t}k_\ell(A^t z) = \sum_{a=0}^{n-1} P^{-a} \sum_{b\in\mathbb{Z}/m_\ell\mathbb{Z}} k_\ell(z+\bar{b})$$
$$= 0,$$

because the inner sum contains one value e_0, one value $-e_0$, and zeros otherwise. Hence the total contribution of k_ℓ to (3) vanishes.

Let

$$N_j = \bigcup_{\ell>j} \bigcup_{t=0}^{nq_j-1} A^{-t}(D_\ell^+ \cup D_\ell^-)$$

and put $F_{j,i} = E_{j,i} \setminus N_j$. If $z \in F_{j,i}$, then none of the points $A^t z$, $0 \le t < nq_j$, belongs to N_∞, and no level above j contributes to (3). The preceding calculation therefore shows that

$$F_{j,i} \subseteq \{\phi^{[q_j]} = e_i\}. \tag{5}$$

The error sets have measure tending to zero. Since $m_{\ell+1} \ge cm_\ell$ and $m_{j+1} = c^2 d_j q_j$, we have

$$\nu(N_j) \le 2nq_j \sum_{\ell>j} \frac{1}{m_\ell} \le 2nq_j \frac{c}{c-1}\frac{1}{m_{j+1}}$$
$$= \frac{2n}{c(c-1)d_j} \longrightarrow 0. \tag{6}$$

To place these returns inside an arbitrary positive-measure set, we need the sets $E_{j,i}$ to become equidistributed over every fixed finite quotient. Let $\mathcal{A}_r$ be the finite σ-algebra generated by π_r. If $r < j$, then $m_r \mid q_j$, and multiplication by n permutes the residue classes modulo m_r. It follows from (4) that every atom of $\mathcal{A}_r$ contains the same proportion $1/c$ of $E_{j,i}$. Thus $\mathbb{E}(\mathbf{1}_{E_{j,i}} \mid \mathcal{A}_r) = 1/c$ whenever $r < j$. Since the union of the spaces $L^2(\mathcal{A}_r)$ is dense in $L^2(Z)$, it follows that $\mathbf{1}_{E_{j,i}} \rightharpoonup 1/c$ weakly in $L^2(Z)$. By (6), the same is true with $F_{j,i}$ in place of $E_{j,i}$.

Finally, B^{q_j} is translation by $\overline{nq_j}$. For each fixed r, one has $m_r \mid q_j$ for all sufficiently large j, so $\overline{nq_j} \to 0$ in Z. The Koopman operators of B^{q_j} therefore converge to the identity on functions that factor through a finite quotient of Z, and hence strongly on $L^2(Z)$. Therefore, for every measurable $E \subseteq Z$,

$$\begin{aligned}\big|\nu(E \cap B^{-q_j}E \cap F_{j,i}) - \nu(E \cap F_{j,i})\big|& \\ \le \|\mathbf{1}_{B^{-q_j}E} - \mathbf{1}_E\|_2 &\longrightarrow 0,\end{aligned}$$

whereas weak convergence gives $\nu(E \cap F_{j,i}) \to \nu(E)/c$. If $\nu(E) > 0$, then $\nu(E \cap B^{-q_j}E \cap F_{j,i}) > 0$ for all sufficiently large j. The conclusion follows from (5). ☐

3. The skew product

The return property turns ergodicity of a fiber action into ergodicity of the associated skew product. Let $\sigma : \Gamma \curvearrowright (Y, \eta)$ be a probability-preserving action, and for $\gamma \in \Gamma$ let $U_\gamma f = f \circ \sigma(\gamma)^{-1}$ be the associated Koopman operator.

Lemma 3.1. *If $\sigma : \Gamma \curvearrowright (Y, \eta)$ is ergodic, then the skew product $T_\sigma(z, y) = \big(Bz, \sigma(\phi(z))y\big)$ on $(Z \times Y, \nu \times \eta)$ is ergodic.*

Proof. Let $F \in L^2(Z \times Y)$ be T_σ-invariant, and write $v_z = F(z, \cdot) \in L^2(Y)$. Then

$$v_{Bz} = U_{\phi(z)} v_z \tag{7}$$

for almost every z. Since Γ is abelian, the function $z \mapsto \|U_{e_i} v_z - v_z\|_2$ is B-invariant for each $i < n$, and is therefore almost everywhere constant.

Fix $i < n$ and $\varepsilon > 0$. The map $z \mapsto v_z$ is strongly measurable, and $L^2(Y)$ is separable. Hence some open ball of radius ε has a positive-measure inverse image $E \subseteq Z$. By Lemma 2.1, there are $q \ge 1$ and a positive-measure set of z such that $z, B^q z \in E$ and $\phi^{[q]}(z) = e_i$. Iterating (7) gives $v_{B^q z} = U_{e_i} v_z$. Thus $\|U_{e_i} v_z - v_z\|_2 < 2\varepsilon$ on a set of positive measure. Since this norm is almost everywhere constant and ε is arbitrary, $U_{e_i} v_z = v_z$ for almost every z. Repeating the argument for all $i < n$, we find that v_z is invariant under σ for almost every z. The action σ is ergodic, so v_z is constant in the Y-variable. Equation (7) and ergodicity of B then imply that F is constant. ☐

For $R \in G$, let $Y = X^n$ with product measure μ^n, and define an action $\rho_R : \Gamma \curvearrowright Y$ by

$$\rho_R(v_0, \dots, v_{n-1}) = R^{v_0} \times \cdots \times R^{v_{n-2}} \times R^{-v_{n-1}}. \tag{8}$$

Thus $\rho_R(e_i)$ acts by R on the ith coordinate for $i < n-1$, while $\rho_R(e_{n-1})$ acts by R^{-1} on the last coordinate. Define

$$\mathcal{G}_n(R)(z, y) = \big(Bz, \rho_R(\phi(z))y\big) \tag{9}$$

on $Z \times Y$.

Proposition 3.2. *If R is ergodic, then $\mathcal{G}_n(R)$ is ergodic. If R is weakly mixing, then the extension $\mathcal{G}_n(R) \to B$ is relatively weakly mixing, and its maximal Kronecker factor is the base system (Z, ν, B).*

Proof. If R is ergodic, then the action ρ_R is ergodic. Indeed, invariance under the coordinate generators removes dependence on the coordinates of Y one at a time. Lemma 3.1 gives ergodicity of $\mathcal{G}_n(R)$.

Now suppose that R is weakly mixing. The relatively independent square of (9) over the base is the skew product associated with the diagonal action $\gamma \mapsto \rho_R(\gamma)\times\rho_R(\gamma)$ on $Y\times Y$. This Γ-action is ergodic. For $i < n-1$, the generator e_i acts by $R \times R$ on the pair of ith coordinates and trivially on the others, while e_{n-1} acts by $R^{-1} \times R^{-1}$ on the last pair. Since these active coordinate transformations are ergodic, invariance under all generators successively removes dependence on every coordinate pair. Lemma 3.1 therefore shows that the relatively independent square is ergodic. This is relative weak mixing of the extension over B.

Let f be a nonzero eigenfunction of $\mathcal{G}_n(R)$. Then the function $F(z, y_0, y_1) = f(z, y_0)\overline{f(z, y_1)}$ is invariant under the relatively independent square and hence is constant almost everywhere. Since $\mathcal{G}_n(R)$ is ergodic, $|f|$ is almost everywhere a positive constant. It follows by Fubini's theorem that $f(z,\cdot)$ is almost everywhere constant for almost every z. Thus every eigenfunction is measurable with respect to the base. Conversely, the characters of Z are eigenfunctions of B and generate its σ-algebra. Hence the maximal Kronecker factor of $\mathcal{G}_n(R)$ is exactly (Z, ν, B). □

Fix a measure-space isomorphism between $Z \times X^n$ and X. By conjugating through this isomorphism, we henceforth regard $\mathcal{G}_n(R)$ as an element of G.

Lemma 3.3. *The map $R \mapsto \mathcal{G}_n(R)$ from G to G is continuous in the weak topology.*

Proof. It is enough to work on $Z \times X^n$. For every fixed $v \in \Gamma$, the map $R \mapsto \rho_R(v)$ is continuous, since G is a topological group and finite products of transformations depend continuously on their coordinates.

Suppose that $R_j \to R$ in G. Let $C \subseteq Z$ and $D \subseteq X^n$ be measurable. From (9),

$$\mathcal{G}_n(R_j)^{-1}(C\times D) = \bigcup_{v\in\Gamma} \big(B^{-1}C \cap \{\phi = v\}\big) \times \rho_{R_j}(v)^{-1}D,$$

where the union is disjoint. Consequently,

$$\begin{aligned}&(\nu\times\mu^n)\big(\mathcal{G}_n(R_j)^{-1}(C\times D) \mathbin{\triangle} \mathcal{G}_n(R)^{-1}(C\times D)\big)\\ &\quad = \sum_{v\in\Gamma} \nu\big(B^{-1}C\cap\{\phi=v\}\big)\mu^n\big(\rho_{R_j}(v)^{-1}D \mathbin{\triangle} \rho_R(v)^{-1}D\big).\end{aligned}$$

Each summand tends to zero, and the sum is dominated by a summable family of total mass at most one. Dominated convergence proves the required convergence for rectangles. Finite unions of rectangles are dense in the measure algebra of $Z \times X^n$, and all the transformations preserve measure, so the same conclusion holds for every measurable set. This is convergence in the weak topology. □

4. Roots and reversibility

Suppose that $U \in G$ satisfies

$$URU^{-1} = R^{-1}. \tag{10}$$

Define $J_U \in \mathrm{Aut}(X^n, \mu^n)$ by

$$\begin{aligned} (J_U x)_0 &= U^{-1} x_{n-1}, \\ (J_U x)_j &= x_{j-1} \qquad (1 \le j \le n-2), \\ (J_U x)_{n-1} &= U x_{n-2}. \end{aligned}$$

When $n = 2$, this reads $J_U(x_0, x_1) = (U^{-1}x_1, Ux_0)$.

Lemma 4.1. *The transformation J_U satisfies $J_U^n = \mathrm{id}$ and*

$$J_U \rho_R(v) J_U^{-1} = \rho_R(Pv) \qquad (v \in \Gamma). \tag{11}$$

Proof. The map J_U moves each coordinate one place forward cyclically. In one circuit, each coordinate is acted on once by U and once by U^{-1}, so $J_U^n = \mathrm{id}$.

It is enough to verify (11) on the standard basis. For $0 \le i < n-2$, the ith and $(i+1)$st coordinate actions are both R, and J_U carries the former to the latter. Equation (10) shows that J_U carries $\rho_R(e_{n-2})$ to $\rho_R(e_{n-1})$. Finally, $U^{-1}R^{-1}U = R$, so J_U carries $\rho_R(e_{n-1})$ to $\rho_R(e_0)$. □

Proposition 4.2. *If $R \cong R^{-1}$, then $\mathcal{G}_n(R)$ has an nth root.*

Proof. Choose U satisfying (10), and define

$$S_U(z, y) = \big(Az, J_U \rho_R(k(z)) y\big). \tag{12}$$

Equation (11) implies $\rho_R(v) J_U^r = J_U^r \rho_R(P^{-r} v)$ for $v \in \Gamma$ and $r \ge 0$. An induction gives

$$S_U^r(z, y) = \left(A^r z, J_U^r \rho_R \left(\sum_{a=0}^{r-1} P^{-a} k(A^a z)\right) y\right)$$

for $1 \le r \le n$. Since $J_U^n = \mathrm{id}$, equations (1) and (9) give $S_U^n = \mathcal{G}_n(R)$. □

Two rigidity facts isolate the obstruction in the converse: B has the unique nth root A on the base, and adjoining an identity factor does not erase conjugacy between ergodic transformations.

Lemma 4.3. *If $C \in \mathrm{Aut}(Z, \nu)$ satisfies $C^n = B$, then $C = A$.*

Proof. The equation $C^n = B$ implies that C commutes with B. Hence $C(z+\overline{n}) = C(z)+\overline{n}$ for almost every z. The measurable function $z \mapsto C(z)-z$ is therefore B-invariant, and so is almost everywhere equal to a constant $c_0 \in Z$. Thus C is translation by c_0. Since $C^n = B$, we have $nc_0 = \overline{n}$. Multiplication by n is injective on Z, and therefore $c_0 = \overline{1}$. Hence $C = A$. □

Lemma 4.4. *Let R and Q be ergodic transformations of (X, μ), and let (W, η) be a standard probability space. If $R \times \mathrm{id}_W \cong Q \times \mathrm{id}_W$, then $R \cong Q$.*

Proof. The invariant σ-algebra of $R \times \mathrm{id}_W$ is the pullback of the σ-algebra of W, and the same is true for $Q \times \mathrm{id}_W$. A conjugacy between the two product transformations therefore induces a measure-space automorphism θ of W. Disintegrating the conjugacy over this factor, and changing it on a null set if necessary, gives a measurable family $U_w \in \mathrm{Aut}(X, \mu)$ such that the conjugacy has the form $(x, w) \mapsto (U_w x, \theta(w))$. The conjugacy equation gives $U_w R = Q U_w$ for almost every w. Thus $R \cong Q$. □

Proposition 4.5. *Let R be weakly mixing. If $\mathcal{G}_n(R)$ has an nth root, then $R \cong R^{-1}$.*

Proof. Suppose that

$$S^n = \mathcal{G}_n(R). \tag{13}$$

Then S commutes with $\mathcal{G}_n(R)$. By Proposition 3.2, the base Z is the maximal Kronecker factor of $\mathcal{G}_n(R)$. A transformation commuting with $\mathcal{G}_n(R)$ preserves the σ-algebra generated by its eigenfunctions. Hence S induces an automorphism C of Z. Equation (13) gives $C^n = B$, and Lemma 4.3 gives $C = A$.

Because S induces A on the factor Z, disintegration over this factor gives a measurable family $V_z \in \mathrm{Aut}(X^n, \mu^n)$ such that $S(z, y) = (Az, V_z y)$ for almost every (z, y). The commutation relation $S\mathcal{G}_n(R) = \mathcal{G}_n(R)S$ then becomes

$$V_{Bz}\rho_R(\phi(z)) = \rho_R(\phi(Az))V_z \tag{14}$$

for almost every z. Let $\widehat{\phi}^{[q]}(z) = \sum_{r=0}^{q-1} \phi(AB^r z)$. Iterating (14) gives

$$V_{B^q z}\rho_R(\phi^{[q]}(z)) = \rho_R(\widehat{\phi}^{[q]}(z))V_z. \tag{15}$$

The definition of ϕ also gives

$$\begin{aligned} \phi(Az) &= \sum_{a=0}^{n-1} P^{-a} k(A^{a+1} z) \\ &= P\phi(z) + P\big(k(Bz) - k(z)\big). \end{aligned}$$

Summing along a B-orbit yields

$$\widehat{\phi}^{[q]}(z) = P\phi^{[q]}(z) + P\big(k(B^q z) - k(z)\big). \tag{16}$$

Choose a B-invariant conull set $Z_0 \subseteq Z$ on which (15) holds for every $q \geq 1$, and put $K_0 = K \cap Z_0$. By Lemma 2.1, $\nu(K_0) > 0$. Choose V in the support of the pushforward of the normalized restriction of ν to K_0 under the map $z \mapsto V_z$. Thus, for every neighborhood $\mathcal{O}$ of V in $\mathrm{Aut}(X^n, \mu^n)$, the set $E_{\mathcal{O}} = \{z \in K_0 : V_z \in \mathcal{O}\}$ has positive measure.

Fix $i < n$. Apply Lemma 2.1 to $E_{\mathcal{O}}$ and e_i. There is $q \geq 1$ such that the set of z satisfying $z, B^q z \in E_{\mathcal{O}}$ and $\phi^{[q]}(z) = e_i$ has positive measure. Since $k(z) = k(B^q z) = 0$, equation (16) gives $\widehat{\phi}^{[q]}(z) = Pe_i$. Equation (15) therefore gives

$$V_{B^q z}\rho_R(e_i) = \rho_R(Pe_i)V_z. \tag{17}$$

Choose a compatible metric on $\mathrm{Aut}(X^n, \mu^n)$ and nested neighborhoods $\mathcal{O}_m$ of V with diameters tending to zero. For each m, choose q_m and z_m as above. Then both V_{z_m} and $V_{B^{q_m} z_m}$ converge to V. Passing to the limit in (17) gives

$$V \rho_R(e_i) V^{-1} = \rho_R(P e_i). \tag{18}$$

Repeating this for every $i < n$ gives (18) for all standard basis vectors.

In particular, $V \rho_R(e_{n-2}) V^{-1} = \rho_R(e_{n-1})$. Up to a permutation of coordinates, the transformations on the two sides before conjugation are respectively $R \times \mathrm{id}_{X^{n-1}}$ and $R^{-1} \times \mathrm{id}_{X^{n-1}}$. Lemma 4.4 gives $R \cong R^{-1}$. □

The following transfer theorem records the weak-mixing domain restriction needed in the final reduction.

Theorem 4.6. *For every weakly mixing $R \in G$, the transformation $\mathcal{G}_n(R)$ is ergodic and*

$$\mathcal{G}_n(R) \text{ has an } n\text{th root} \iff R \cong R^{-1}.$$

Moreover, the map $R \mapsto \mathcal{G}_n(R)$ is continuous in the weak topology.

Proof. Ergodicity and continuity follow from Proposition 3.2 and Lemma 3.3. The two implications are Propositions 4.2 and 4.5. □

5. Proof of the main theorem

Let Trees be the Polish space of trees, and let $\mathsf{IF} \subseteq \mathsf{Trees}$ be the set of ill-founded trees (i.e., trees that have an infinite branch). The set IF is $\mathbf{\Sigma}^1_1$-complete; see [7, Section 27] and [9, Section 2.5].

Fix one of the compact surfaces appearing in [9, Theorem 3], for example the two-torus. Kunde constructs a continuous one-to-one map $\tau \mapsto R_\tau$ from Trees into the weakly mixing, area-preserving C^∞ diffeomorphisms of that surface such that

$$\tau \in \mathsf{IF} \iff R_\tau \cong R_\tau^{-1}. \tag{19}$$

Indeed, the forward implication is part (1) of [9, Theorem 3]; part (2) states that Kakutani equivalence of R_τ and R_τ^{-1} implies that τ is ill-founded, and hence gives the reverse implication in (19). The inclusion of the C^∞ diffeomorphism group into the group of measure-preserving transformations is continuous from the C^∞ topology to the weak topology: C^∞ convergence, together with continuity of inversion, gives uniform convergence of the associated Koopman operators on continuous functions, and continuous functions are dense in L^2. After conjugating by a fixed measure-space isomorphism with (X, μ), we may therefore regard $\tau \mapsto R_\tau$ as a continuous map into G.

Proof of Theorem 1.1. Fix $n \geq 2$ and define $\Theta_n : \mathsf{Trees} \to G$ by $\Theta_n(\tau) = \mathcal{G}_n(R_\tau)$. By Lemma 3.3, Θ_n is continuous. Every R_τ is weakly mixing, so

Theorem 4.6 gives

$$\begin{aligned} \tau \in \mathsf{IF} \quad &\Longleftrightarrow \quad R_\tau \cong R_\tau^{-1} \\ &\Longleftrightarrow \quad \Theta_n(\tau) \text{ has an } n\text{th root.} \end{aligned}$$

Every $\Theta_n(\tau)$ is ergodic. Thus Θ_n is a continuous reduction of IF to $\mathcal{R}_n^{\mathrm{erg}}$. Since $\mathcal{R}_n^{\mathrm{erg}}$ is analytic, it is $\mathbf{\Sigma}_1^1$-complete. □

Proof of Corollary 1.2. The same map Θ_n is a continuous reduction of IF to $\mathcal{R}_n$, because all of its values are ergodic. Hence $\mathcal{R}_n$ is $\mathbf{\Sigma}_1^1$-hard. It is analytic as the range of the continuous power map $S \mapsto S^n$, so it is $\mathbf{\Sigma}_1^1$-complete. □

6. Further questions

The reduction above has weakly mixing inputs but not weakly mixing outputs: every $\Theta_n(\tau)$ has the nontrivial odometer (Z, ν, B) as its maximal Kronecker factor. Is the set $\mathcal{R}_n^{\mathrm{wm}} = \{T \in \mathcal{R}_n : T \text{ is weakly mixing}\}$ $\mathbf{\Sigma}_1^1$-complete for every $n \geq 2$?

Theorem 1.1 treats every fixed root order uniformly, so at this level there is no distinction between prime and composite n. Arithmetic enters when several root orders are required of the same transformation: if T has an nth root, then it has a dth root for every divisor d of n. For a fixed finite or infinite set $D \subseteq \{2, 3, \ldots\}$, what is the descriptive complexity of $\bigcap_{m \in D} \mathcal{R}_m^{\mathrm{erg}}$? More finely, which downward-closed sets occur as $\{m \geq 2 : T \in \mathcal{R}_m\}$, and what is the complexity of prescribing this root spectrum? King proved that a generic transformation has roots of every order [8], while Madore constructed a mixing transformation with square-root chains of every finite length but no infinite square-root chain [10].

Madore's example also separates independent roots from a coherent system of roots. Call a family $(S_m)_{m \geq 1}$ coherent if $S_1 = T$ and $S_{mn}^m = S_n$ for all $m, n \geq 1$; equivalently, it determines a measure-preserving action of $(\mathbb{Q}, +)$ whose time-one map is T. What is the descriptive complexity of the transformations admitting such a rational action? A time-one map of a measure-preserving flow has a coherent rational root system, but flow embeddability requires that this action extend to a weakly continuous one-parameter subgroup. De la Rue and de Sam Lazaro proved that a generic measure-preserving transformation embeds in a flow [2]. What is the exact descriptive complexity of the time-one maps of measure-preserving flows, and does it change upon restriction to the ergodic or weakly mixing transformations?

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