

FINITE-INDEX EXTENSIONS OF ESSENTIALLY FREE COUNTABLE BOREL EQUIVALENCE RELATIONS

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ABSTRACT. Kechris asked whether essential freeness of countable Borel equivalence relations is preserved under extensions of finite index. Kaya proved that every such extension of a relation E is Borel reducible to the equivalence relation E^{fin} on nonempty finite subsets which records the E -classes met by the subset. We show that E^{fin} is essentially free whenever E is. This gives an affirmative answer to Kechris's question.

1. INTRODUCTION

A countable Borel equivalence relation E on a standard Borel space X is *free* if it is the orbit equivalence relation of a free Borel action of a countable group. We use the equivalent characterization that E is *essentially free* if it is Borel reducible to a free countable Borel equivalence relation; see [5, Chapter 11].

Let $E \subseteq F$ be countable Borel equivalence relations on the same standard Borel space. Following Kaya [4, Section 4], write $[F : E] < \infty$ if every F -class is the union of finitely many E -classes; no uniform bound is assumed. Kechris asks whether

$$[F : E] < \infty \quad \text{and} \quad E \text{ essentially free} \quad \implies \quad F \text{ essentially free}$$

[5, Problem 11.4]. We prove that the answer is yes.

Theorem 1. *Let $E \subseteq F$ be countable Borel equivalence relations. If $[F : E] < \infty$ and E is essentially free, then F is essentially free.*

2. FINITE SETS OF EQUIVALENCE CLASSES

For a standard Borel space X , let $\text{Fin}(X)$ denote the standard Borel space of nonempty finite subsets of X . If E is a Borel equivalence relation on X , define E^{fin} on $\text{Fin}(X)$ by

$$a E^{\text{fin}} b \iff \{[x]_E : x \in a\} = \{[y]_E : y \in b\}.$$

Thus two finite subsets are equivalent exactly when they meet the same collection of E -classes. Kaya considered this finite-subset variant of the Friedman–Stanley jump and proved the following [4, Proposition 11].

Proposition 2 (Kaya). *Let $E \subseteq F$ be countable Borel equivalence relations on X . If $[F : E] < \infty$, then*

$$F \leq_B E^{\text{fin}}.$$

Proof. For completeness, we recall Kaya's argument. By the Feldman–Moore theorem [2], there is a Borel action of a countable group G on X whose orbit equivalence relation is F . Enumerate $G = \{g_i : i \in \mathbb{N}\}$ with $g_0 = 1_G$. For $x \in X$, let $j(x)$ be the least j such that

$$(\forall k \in \mathbb{N})(\exists i \leq j) \quad g_k \cdot x E g_i \cdot x.$$

Such a j exists because $[x]_F$ contains only finitely many E -classes, and the map $x \mapsto j(x)$ is Borel. Set

$$\phi(x) = \{g_i \cdot x : i \leq j(x)\}.$$

The set $\phi(x)$ meets every E -class contained in $[x]_F$, and it meets no other E -class. Consequently,

$$x F y \iff \phi(x) E^{\text{fin}} \phi(y),$$

so ϕ is a Borel reduction. $\square$

The remaining observation is that essential freeness is preserved by $E \mapsto E^{\text{fin}}$.

Proposition 3. *If E is an essentially free countable Borel equivalence relation, then E^{fin} is essentially free.*

Proof. Suppose first that E is induced by a free Borel action $\Gamma \curvearrowright X$. For $n \geq 1$, let

$$D_n = \{(x_0, \dots, x_{n-1}) \in X^n : i \neq j \implies \neg(x_i E x_j)\},$$

and define a countable Borel equivalence relation R_n on D_n by

$$(x_i)_{i < n} R_n (y_i)_{i < n} \iff (\exists \sigma \in \mathfrak{S}_n)(\forall i < n) \quad x_i E y_{\sigma(i)}.$$

The wreath product $\Gamma^n \rtimes \mathfrak{S}_n$ acts on D_n by

$$((\gamma_i)_{i < n}, \sigma) \cdot (x_i)_{i < n} = (\gamma_i \cdot x_{\sigma^{-1}(i)})_{i < n},$$

and its orbit equivalence relation is R_n . This action is free. Indeed, if the displayed group element fixes $(x_i)_{i < n}$, then $x_i E x_{\sigma^{-1}(i)}$ for every i . Since the coordinates lie in pairwise distinct E -classes, σ is the identity; freeness of the original action then gives $\gamma_i = 1_\Gamma$ for every i .

It follows from the closure of essential freeness under countable disjoint sums [5, Proposition 11.3(ii)] that

$$\bigoplus_{n \geq 1} R_n$$

is essentially free. Fix a Borel linear order $<$ on X . For $a \in \text{Fin}(X)$, put

$$c(a) = \{x \in a : \neg(\exists y \in a)(y < x \wedge y E x)\}.$$

Thus $c(a)$ contains the least member of a from each E -class met by a . The map $a \mapsto c(a)$ is Borel, and listing the members of $c(a)$ in increasing order gives a Borel reduction

$$E^{\text{fin}} \leq_B \bigoplus_{n \geq 1} R_n.$$

Therefore E^{fin} is essentially free.

Now let E be merely essentially free, and let f be a Borel reduction of E to a free countable Borel equivalence relation Q . As Kaya also notes [4, Section 4], the map

$$a \mapsto f[a]$$

reduces E^{fin} to Q^{fin} . The preceding paragraph shows that Q^{fin} is essentially free, and hence so is E^{fin} . $\square$

Proof of Theorem 1. By Proposition 2, $F \leq_B E^{\text{fin}}$. By Proposition 3, E^{fin} is essentially free. Therefore F is essentially free. $\square$

3. REMARKS

Remark 4. The proof does not settle the analogous question for treeability [3, Section 6.4(B)]. The point used above is that the relevant wreath-product actions remain free. The same construction gives no treeing of their orbit equivalence relations.

Remark 5. The finiteness assumption in Theorem 1 cannot be replaced by countable index. Let M be a countable transitive model of a sufficiently large finite fragment of ZFC, let $\text{Rand}(M)$ be the set of reals random over M , and put

$$x E_M^{\mathbb{B}} y \iff M[x] = M[y].$$

The relation $E_0 \upharpoonright \text{Rand}(M)$ is free and is a subrelation of $E_M^{\mathbb{B}}$. It has countably infinite index in every $E_M^{\mathbb{B}}$ -class. Indeed, for $x \in \text{Rand}(M)$, the tails

$$x^{(n)}(k) = x(k + n)$$

are all random over M , satisfy $M[x^{(n)}] = M[x]$, and are pairwise E_0 -inequivalent; otherwise x would be eventually periodic. On the other hand, $E_M^{\mathbb{B}}$ is not essentially free [1, Corollary 1.2].

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