Product Forcing, Iterated Forcing, Continuum Coding, and Forcing the Ground Axiom

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Recall GA is the Ground Axiom stating that the universe is not a (nontrivial) forcing extension of some inner model. We'll look at consistency results about GA. Eventually, we will see that $\text{Con}(\mathsf{ZFC}) \to \text{Con}(GA)$. We show this by noting that $\text{Con}(\mathsf{ZFC}) \to \text{Con}(GCH)$ and $\text{Con}(GCH) \to \text{Con}(CCA)$, where CCA is an axiom we'll introduce, and finally $\mathsf{ZFC} \vdash CCA \to GA$. We shall look at

1. a general fact about models of ZFC
2. how that fact is relevant to our topic
3. generalizations of a forcing poset we've seen
4. the continuum coding axiom and its relation to GA
5. how to force infinitely many times using product forcing and iterated forcing
6. how to use iterated (class) forcing to obtain a model satisfying the CCA and hence GA.

A general fact about transitive models of ZFC is that they are entirely determined by what sets of ordinals they have (recall that under ZFC, every set is coded by a set of ordinals).

Proposition 1. Let M, N be transitive models of ZF with the same ordinals, such that for every α, $\mathcal{P}^M(\alpha) = \mathcal{P}^N(\alpha)$. Let at least one of them satisfy Choice. Then $M = N$

Proof. Without loss of generality, suppose M satisfies AC. Since Göddel's pairing function $\Gamma : ON \times ON \to ON$ is absolute, we note that for every ordinal α, we have $\mathcal{P}(\alpha \times \alpha)^M = \mathcal{P}(\alpha \times \alpha)^N$ as well.

First we show $M \subseteq N$. Let $X \in M$. By $(AC)^M$, we see that there is some $f \in M$ such that $f : \theta \leftrightarrow \text{tr cl}(\{X\})$ for some ordinal $\theta \in M$. Now we define the relation E on θ as follows: $\alpha E \beta \iff f(\alpha) \in f(\beta)$. Now E is just a set of ordered pairs of ordinals in M, so by assumption $E \in N$. In N, we may take the Mostowski collapse of (θ, E) and get an isomorphic $(T, \in)$. But notice that this is just $\text{tr cl}(\{X\})$. Now since $X \in \text{tr cl}(\{X\}) \in N$ and N is transitive, we have $X \in N$.

Now we show $N \subseteq M$. We will prove this by $\in$-induction. Let $X \in N$ and suppose that the claim holds for every $x \in X$ (i.e., $X \subseteq M$). Take $Y \in M$ such that $X \subseteq Y$. So we consider $f \in M$ which well-orders Y. Since $M \subseteq N$, this f will be in N and hence $f(X) \in N$. But $f(X) \subseteq ON$, so $f(X) \in M$. Taking the reverse image, we have that $X \in M$. $\square$

So to control whether two models of ZFC are identical, it is sufficient to control what sets of ordinals they have. Let $CH(\kappa)$ be the following statement: $2^\kappa = \kappa^+$. Also recall that every cardinal κ is of the form $\aleph_\alpha$, where α is an ordinal. So we may view $CH(\aleph_\alpha)$ as a statement/property about α. Controlling whether a model of ZFC satisfies $CH(\aleph_\alpha)$ or not gives us a way to control this property of α. Let's look at a naive toy example.

Example 2. Let $M \vDash \mathsf{ZFC} + GCH$ be a ctm. Consider whether $CH(\aleph_\alpha)$ holds for $\omega \leq \alpha \leq \omega_1$. We may consider a "characteristic function" $f : [\omega, \omega_1] \to \{0, 1\}$ marking whether $CH(\aleph_\alpha)$ holds. Since M satisfies GCH, we have that f is a constant function mapping everything to 1.

But we can use forcing to change that. Say we force such that $M[G] \vDash 2^{\aleph_{\omega+4}} = \aleph_{\omega+7}$. Then in $M[G]$, $f(4) = 0$. Assuming all else are unchanged, we may think of f as a characteristic function of the set $\{\alpha \mid \omega \leq \alpha \leq \omega_1 \wedge \alpha \neq 4\}$. Manipulating this and other "characteristic functions", we can code sets of ordinals with forcing to change the continuum pattern.

The example above leads to the following definition:

Definition 3. Recall that if x is a set, then there is some ordinal δ_x and relation $E_x \subseteq \delta \times \delta$ such that $(tc(\{x\}), \in) \cong (\delta_x, E_x)$ Let α, λ be ordinals. Let $g : ON \to ON \times ON$ be the inverse of Gödel's pairing function. Define the set $c(\alpha, \lambda) \subseteq \lambda$ as follows: for all $i \in \lambda$, $i \in c(\alpha, \lambda) \iff 2^{\aleph_{\alpha+i+1}} = \aleph_{\alpha+i+2}$. We say that *$x$ is coded into the continuum pattern at α with length λ* iff $g''c(\alpha, \lambda) = E_x$. That is, $\in \restriction tc(\{x\}) \cong g''c(\alpha, \lambda)$.

We say that *x is coded into the continuum pattern* when there is some α, λ such that x is coded into the continuum pattern at α with length λ.

Remark. Note that we are coding i by the behavior of the continuum function at $\aleph_{\alpha+i+1}$ instead of $\aleph_{\alpha+i}$. This is because we have greater forcing control over regular cardinals than singular cardinals, and all successor cardinals are regular.

To control the continuum function $\kappa \mapsto 2^\kappa$ at regular cardinals, we use a generalization of the poset $\mathcal{P} = (2^{[\omega \times \omega_2]^{<\omega}}, \supseteq)$. We now study the properties of this type of posets.

Definition 4. Let κ, λ be cardinals, let $\mathrm{Add}(\kappa, \lambda)$ the set $\{p : p$ is a partial function $\kappa \times \lambda \to 2\&|p| < \kappa\}$, ordered by $\supseteq$. That is, $p \leq q \iff p \supseteq q$ (p is stronger than q, in the sense of containing more information). Intuitively, $\mathrm{Add}(\kappa, \lambda)$ reads "add to κ λ-many subsets". The poset to force $\neg CH$ we've seen is $\mathrm{Add}(\omega, \omega_2)$.

As one would expect, forcing with $\mathrm{Add}(\kappa, \lambda)$ will give us $M[G] \vDash 2^\kappa \geq \lambda$. The precise derivation of this fact comes from the following propositions:

Proposition 5. $\mathrm{Add}(\kappa, \lambda)$ has the $(2^{<\kappa})^+$-cc.

Proof. Let $\mu = (2^{<\kappa})^+$. Suppose for contradiction that $A \subseteq \mathrm{Add}(\kappa, \lambda)$ is an antichain with cardinality μ. We first deal with the case where κ is regular.

Let $\langle p_\xi \mid \xi < \mu \rangle$ enumerate A. Let $S_i = \operatorname{dom} p_i$. By the delta system lemma, we get $D \subseteq \mu$ such that $\mathrm{card}(D) = \mu$ so that $\{S_i \mid i \in D\}$ forms a delta system with root R. Since $|R| < \kappa$ (why?) we have that $2^| R| < \mu$.

Claim: there must be $i \neq j \in D$ such that p_i and p_j are compatible. If this claim holds, then we have a contradiction to the assumption that A is an antichain.

Proof of claim: For $\{p_i \mid i \in D\}$ to be pairwise incompatible, we need that $\{p_i \restriction R \mid i \in D\}$ to be pairwise disjoint. Note that each $p_i \restriction R$ is a $0-1$ sequence of length $|R|$. But the total number of length-$|R|$ $0-1$ sequences is $2^{|R|} < \mu = |D|$. By the pigeonhole principle, there must be $i \neq j$

such that $R = \mathrm{dom}(p_i) \cap \mathrm{dom}(p_j)$ and $p_i \restriction R = p_j \restriction R$. But then $q = p_i \cup p_j$ is a common extension to them. This finishes the proof of claim and the case for κ regular.

If κ were singular, then we note that $A = \bigcup_{\lambda<\kappa}\{p_\xi \in A \mid |p_\xi| < \lambda\}$. But the cardinality of A is μ, a regular cardinal. Then there must be some $\kappa_0 < \kappa$ such that $\{p_\xi \in A \mid |p_\xi| < \kappa_0\} = A'$ has size μ. Then we may proceed with the above proof by considering A' instead, and prove that A' can't be an antichain.

□

Proposition 6. If $\mathbb{P}$ has the μ-cc, then forcing with $\mathbb{P}$ preserves cofinalities and cardinals $\geq \mu$ (exercise, straightforward generalization of the ccc case)

Proof. exercise. straightforward generalization with the case of ccc. □

Remark. Note that this proposition also tells us that if $\mathbb{P}$ is a forcing poset, then the cardinals $> |\mathbb{P}|$ are preserved (because trivially $\mathbb{P}$ has the $|\mathbb{P}|^+$-cc).

Here is a notion that allows us to show preservations of small cofinalities and cardinalities.

Definition 7. $\mathbb{P}$ is κ-closed, iff given any descending sequence $(p_i \mid i < \alpha)$, where $\alpha < \kappa$, there is a lower bound for that sequence.

Proposition 8. If $\mathbb{P}$ is κ-closed, the forcing with $\mathbb{P}$ preserves cofinalities and cardinalities $\leq \kappa$.

Proof. First I claim that it suffices to show $\mathbb{P}$ preserves regular cardinals $\leq \kappa$. Why? Recall that $\mathrm{cof}^M(\alpha) \geq \mathrm{cof}^{M[G]}(\alpha)$ for all α. For the other direction, let $\mathrm{cof}^M(\alpha) = \gamma$. Note that cofinalities can only be regular cardinals and since $\mathbb{P}$ preserves regular cardinal, γ is also a regular cardinal in $M[G]$. Suppose for contradiction that $\mathrm{cof}^M(\alpha) > \mathrm{cof}^{M[G]}$. Then there is in $M[G]$ cofinal functions $f : \gamma \to \alpha$ and $g : \delta \to \alpha$ where $\delta < \gamma$. Now we will construct in $M[G]$ a cofinal function $h : \delta \to \gamma$, contradicting γ's regularity in $M[G]$. We define h as follows: $h(i) = \min\{j \in \gamma \mid g(i) < f(j)\}$. We see that h is well-defined (picture?), and maps δ cofinally to γ.

To see that a κ-closed $\mathbb{P}$ preserves regular cardinals $\leq \kappa$, let λ be a regular cardinal $\leq \kappa$, and let $G \subseteq \mathbb{P}$ be M-generic. Suppose for contradiction that $M[G] \vDash \exists f : \delta \to_{\mathrm{cof}} \lambda$ where $\delta < \lambda$. I will show that such an f is in M as well, contradicting λ's regularity in M.

To see this, fix $p_0 \in G$ with $p_0 \Vdash \dot{f} : \check{\delta} \to_{\mathrm{cof}} \check{\lambda}$. And for $i < \delta$, fix $p_{i+1} \leq p_i$ such that p_{i+1} decides that value of f on i (that is, for some $\alpha \in \lambda$, $p_{i+1} \Vdash \dot{f}(\check{i}) = \check{\alpha}$). And if $\mu < \delta$ is a limit ordinal, then by κ-closure, we may fix a lower bound p_μ to the sequence $\langle p_i \mid i < \mu\rangle$. To see that this lower bound is in G, we notice that $D = \{q \in \mathbb{P} \mid \forall i < \mu \; q \leq p_i \mathrm{or} \exists i < \mu (q \perp p_i)\}$ is dense. So $G \cap D \neq \emptyset$. But then the only possibility is that $G \cap D$ contains some lower bound of that sequence.

Finally, let $p_\delta \in G$ be a lower bound to the sequence $\langle p_i \mid i < \delta\rangle$. It follows that p_δ is in M and p_δ decides the value of $\dot{f}(\check{i})$ for all $i < \delta$. So we can define f in M as $f(i) = \alpha$ iff $p_\delta \Vdash \dot{f}(\check{i}) = \check{\alpha}$. So $f \in M$, contradiction. □

Proposition 9. Suppose κ is regular, then $\mathrm{Add}(\kappa, \lambda)$ is κ-closed

Proof. let $\langle p_i \mid i < \mu\rangle$ be a sequence of elements from $\mathrm{Add}(\kappa, \lambda)$ and $\mu < \kappa$. Now let $p = \bigcup_{i<\mu} p_i$. Now p is a partial function from $\kappa \times \lambda$ to $\{0, 1\}$. Moreover, each p_i has cardinality $< \kappa$ by definition, and we are unioning $< \kappa$ many of them, so by κ's regularity, we have that $|p| < \kappa$. So $p \in \mathrm{Add}(\kappa, \lambda)$. □

Here we define a concept that will help us obtain an upper bound of 2^κ in the forcing extension, which we will need later in some arguments.

Definition 10. Let τ be a $\mathbb{P}$-name. Then a *nice name* of a subset of τ is a $\mathbb{P}$-name of the form

$$\bigcup\{\{\pi\} \times A_\pi \mid \pi \in \mathrm{dom}(\tau)\}$$

where each A_π is an antichain in $\mathbb{P}$.

The following series of propositions will illustrate how the nice names place an upper bound on the continuum function.

Proposition 11. Let $\mathbb{P}$ be a poset such that $\kappa = |\mathbb{P}|$ and that $\mathbb{P}$ has the λ-cc. Let τ be a $\mathbb{P}$-name with $|\tau| = \mu$. Then there are at most $(\kappa^{<\lambda})^\mu$ many nice names for subsets of τ.

Proof. By λ-cc, each A_σ has cardinality $< \lambda$, so $\mathbb{P}$ has at most $\kappa^{<\lambda}$ many antichains. And since σ will have to come from $\mathrm{dom}(\tau)$ and $|\tau| = \mu$, there can only be at most $(\kappa^{<\lambda})^\mu$ many nice names. $\square$

Proposition 12 (AC)**.** If σ, τ are $\mathbb{P}$-names, then there is a nice name ρ such that $1 \Vdash \sigma \subseteq \tau \to \sigma = \rho$. (More snappily: every subset of τ is given by a nice name).

Proof. Given τ, we define $\rho := \bigcup\{\{\pi\} \times A_\pi \mid \pi \in \mathrm{dom}(\tau)\}$, where A_π satisfies:

- A_π is an antichain
- each $p \in A_\pi$ is such that $p \Vdash \pi \in \sigma$
- for any B with the above two properties, $B \subseteq A_\pi$

The existence of such A_π's is guaranteed by Zorn's lemma.

Now suppose $G \subseteq \mathbb{P}$ is M-generic and $M[G] \vDash \sigma_G \subseteq \tau_G$. We show that $\sigma_G = \tau_G$.
$\supseteq$: let $a \in \rho_G$, then by definition of ρ, there is some $\pi \in \mathrm{dom}(\tau)$ such that $a = \pi_G$, and there is some $p \in G$ with $p \Vdash \pi \in \sigma$. This means that $a = \pi_G \in \rho_G$.
$\subseteq$: Let $a \in \sigma_G$. It follows that $a = \pi_G$ for some $\pi \in \mathrm{dom}(\tau)$ (we are assuming $\sigma_G \subseteq \tau_G$). For this π, if $A_\pi \cap G \neq \emptyset$, then any $p \in A_\pi \cap G$ will have $\langle \pi, p\rangle \in \rho$, and since $p \in G$, we have $\pi_G = a \in \rho_G$. On the other hand, if $A_\pi \cap G = \emptyset$, then $D_\pi := \{p : \exists r \in A_\pi (p \leq r)\} \cup \{p : \forall r \in A_\pi (p \perp r)\}$ is dense (exercise). So $G \cap D_\pi \neq \emptyset$. And since $A_\pi \cap D = \emptyset$, the only possibility is that some $q \in G \cap D \subseteq \{p : \forall r \in A_\pi (p \perp r)\}$. Let $q' \in G$ be such that $q' \Vdash \pi \in \sigma$ and let $r \in G$ be a common extension to them both. Then $A_\pi \cup \{r\}$ satisfies the first two bullet points above, contradicting the maximality of A_π. $\square$

We can now compute upper bounds using nice names

Proposition 13. Suppose $\mathbb{P}$ is a poset of cardinality κ, and suppose also $\mathbb{P}$ has the λ-cc. Let μ be a cardinal and let $\delta = (\kappa^{<\lambda})^\mu$. Then, if $G \subseteq \mathbb{P}$ is M-generic, then $M[G] \vDash 2^\mu \leq \delta$

Proof. the name $\check{\mu}$ has size μ. So there are at most δ many nice names for subsets of $\check{\mu}$. Let $\langle \rho_x i \mid \xi < \delta\rangle$ enumerate them. Let $\dot{f} = \{(\mathrm{op}(\check{\xi}, \rho_\xi), 1) : \xi < \delta\}$. Then in $M[G]$ we have that f_G is a surjective function with domain δ and range $\mathcal{P}(\lambda)$. Hence in $M[G]$, f witnesses $2^\mu \leq \delta$. $\square$

Why do we care about upper bounds of the continuum function? The significance of the preceding (i.e., limitations of set-sized forcings) is illustrated in the following proposition:

Proposition 14. For a set-sized forcing poset $\mathbb{P}$, forcing with $\mathbb{P}$ can only affect the continuum function on an initial segment.

Proof. say $|\mathbb{P}| = \kappa$, then trivially $\mathbb{P}$ has the κ^+-cc. So cardinals $> \kappa$ are preserved.

Now recall that if μ is a cardinal, then there are at most $(\kappa^\kappa)^\mu$ many nice names for subsets of μ (because $|\check{\mu}| = \mu$). Take μ to be sufficiently large (for definiteness, $\mu = \kappa$ suffices), we have $(\kappa^\kappa)^\mu = 2^\mu$. This means that $\mathcal{P}^{M[G]}(\mu)$ can have at most $(2^\mu)^M$ many elements. And because no cardinals $> \kappa$ are collapsed, we have that $(2^\mu)^{M[G]} \geq (2^\mu)^M$. □

Remark. In fact the above proof shows: If $\mathbb{P}$ is a forcing poset, then cardinals and continuum patterns $|\mathbb{P}|$ are preserved.

Recall the notion of coding into the continuum pattern:

Definition 3. Recall that if x is a set, then there is some ordinal δ_x and relation $E_x \subseteq \delta \times \delta$ such that $(tc(\{x\}), \in) \cong (\delta_x, E_x)$ Let α, λ be ordinals. Let $g : ON \to ON \times ON$ be the inverse of Gödel's pairing function. Define the set $c(\alpha, \lambda) \subseteq \lambda$ as follows: for all $i \in \lambda$, $i \in c(\alpha, \lambda) \iff 2^{\aleph_{\alpha+i+1}} = \aleph_{\alpha+i+2}$. We say that x *is coded into the continuum pattern at* α *with length* λ iff $g''c(\alpha, \lambda) = E_x$. That is, $\in\restriction tc(\{x\}) \cong g''c(\alpha, \lambda)$.

We say that x *is coded into the continuum pattern* when there is some α, λ such that x is coded into the continuum pattern at α with length λ.

We are now ready to introduce an axiom that implies the Ground Axiom.

Definition 15 (Definition 3.1 in Reitz)**.** The continuum Coding Axiom (CCA) is the assertion that for every ordinal α and every $a \subseteq \alpha$, there is an ordinal θ such that $\beta \in a \leftrightarrow 2^{\aleph_{\theta+\beta+1}} = (\aleph_{\theta+\beta+1})^+$ for every $\beta < \alpha$.

In our terminology, this axiom says every set of ordinals is coded into the continuum pattern.

Theorem 16. CCA implies GA.

Proof. suppose $V \vDash CCA$ and that $V = W[h]$ is a set-forcing extension of W, obtained from the poset $\mathbb{Q} \in W$, where $h \subseteq \mathbb{Q}$ is W-generic. By our previous results, cardinals and continuum patterns $> |\mathbb{Q}|$ are preserved.

Now every set of ordinals is coded into the continuum pattern in V. It suffices to show that one such code occurs above $|\mathbb{Q}|$. Why? Because cardinal and continuum stuff above $|\mathbb{Q}|$ is not affected. So one such code will be in W. So V and W have the same sets of ordinals, and hence they are identical.

To see that one such code occurs above $|\mathbb{Q}|$, let $|\mathbb{Q}| = \aleph_\delta$ and consider the set $a' = \{\delta+\beta \mid \beta \in a\}$. Now $a' \in V$, so a' is coded into the continuum pattern. But notice how the codes of a' is just a copy of the codes of a, and shifted above $\aleph_\delta = |\mathbb{Q}|$. Hence a is coded in the continuum pattern above $|\mathbb{Q}|$ as well. □

We want to know if we can obtain a model satisfying CCA. Doing this requires manipulating the continuum function many, many times. We've seen that if $M \vDash ZFC$ then $M[G] \vDash ZFC$. But then we can carry out forcing arguments in $M[G]$ as well, we get, say $M[G][H]$. In fact, we can do this finitely many times, but we need to be careful if we want to do this transfinitely many times.

Remark. There exists a chain of models $M_0 \subseteq M_1 \subseteq M_2 \subseteq ...$ such that $M_0 \vDash ZFC$ is a ground model and each M_{n+1} is a generic extension of M_n, but that $\bigcup_{n\in\omega} M_n$ is not a model of ZFC.

To see this, take $M_0 \vDash \mathsf{ZFC}$ to be a ctm. Since M_0 is countable, we may fix (in V) a cofinal sequence $\langle \kappa_n \mid n \in \omega\rangle$ of the cardinals in M_0. Let M_{n+1} be $M_n[G_n]$ where $G_n \subseteq (Add(\omega, \kappa_n))^{M_n}$ is a generic filter. This forces $(2^\omega)^{M_{n+1}}$ to be bigger than κ_n. At the limit, we are going to have that 2^ω is greater than all cardinals. So powerset axiom fails.

So some care is needed to manipulate the continuum pattern infinitely many times. We may ask, for example, can we ask a single forcing poset to do to the continuum pattern than just altering one place of it? Here's a (perhaps badly drawn) analogy to motivate the construction:

We've seen how to add a single real with the poset $(\{p : p \text{ is a partial function } \omega \to 2 \wedge |p| < \omega\}, \supseteq)$. To add ω_2-many reals, we didn't force ω_2-many times with that poset. Instead, we let the conditions in our poset have product domains ($p : \omega \times \omega_2 \to 2$). This way, plus some density arguments, we managed to add ω_2-many reals with one poset. Can we do something like this to force infinitely many times (each time with some single $\text{Add}(\kappa, \lambda)$), but with one poset? The hint from our analogy is that we might want to use some kind of product construction.

Definition 17. Let $\mathbb{P}$, $\mathbb{Q}$ be posets. Let the *product poset* be ($\mathbb{P} \times \mathbb{Q}$, with an ordering relation inherited point-wise from the factors: $(p, q) \leq (p' \leq q') \iff p \leq p' \wedge q \leq q'$.

Proposition 18. $\mathbb{P} \times \mathbb{Q}$ is a poset and $1_{\mathbb{P}\times\mathbb{Q}} = (1_\mathbb{P}, 1_\mathbb{Q})$.

Proof. exercise. □

Now that we have our product poset. We can study its generic filters.

Proposition 19. Suppose $K \subseteq \mathbb{P} \times \mathbb{Q}$ is M-generic. Let $G = \{p \in \mathbb{P} \mid \exists q\ (p, q) \in K\}$ and $H = \{q \in \mathbb{Q} \mid \exists p(p, q) \in K\}$. Then $G \subseteq \mathbb{P}$ and $H \subseteq \mathbb{Q}$ are M-generic and $K = G \times H$.

This proposition follows from a more general fact. To state that fact, we first need a definition.

Definition 20. Let A, B be posets. A function $e : A \to B$ is a *complete embedding* iff

1. $e(1_A) = 1_B$
2. $a \leq a' \Rightarrow e(a) \leq e(a')$
3. $a \perp a' \Rightarrow e(a) \perp e(a')$
4. If $C \subseteq A$ is a maximal antichain, then $e''A$ is a maximal antichain.

Proposition 21. Let $e_0 : \mathbb{P} \to \mathbb{P}\times\mathbb{Q}$ and $e_1 : \mathbb{Q} \to \mathbb{P}\times\mathbb{Q}$ be complete embeddings. Then any generic $K \subseteq \mathbb{P} \times \mathbb{Q}$ determines generic $G \subseteq \mathbb{Q}, H \subseteq \mathbb{Q}$. Namely $G = e_0^{-1}[K] = \{p \in \mathbb{P} \mid \exists q \in \mathbb{Q}\ e_0(p) \in K\}$ and $H = e_1^{-1}[K] = \{q \in \mathbb{Q} \mid \exists p \in \mathbb{Q}\ e_1(q) \in K\}$. Moreover, $K = G \times H$

Proof. That G and H are filters are left as exercises. We show that they are dense. Let $D \subseteq \mathbb{P}$ be an arbitrary dense subset. Then $D \times \mathbb{Q}$ is a dense subset of $\mathbb{P} \times \mathbb{Q}$ (given (p, q), find in D some $p' \leq p$, so $(p', q) \leq (p, q)$). Hence $K \cap (D \times \mathbb{Q}) \neq \emptyset$. This means that $(e_0^{-1}[K] = G) \cap D \neq \emptyset$. Argue similarly for H.

For the moreoever: $K \subseteq G \times H$ by definition (why?). To see $G \times H \subseteq K$: take $p \in G$ and $q \in H$. Then $(p, 1_\mathbb{Q}), (1_\mathbb{P}, q) \in K$. Since K is a filter (hence downward directed), we may take $(p', q') \in K$ below them. By upward closure, $(p', q') \leq (p, q)$ implies $(p, q) \in K$. □

Lemma 22. the embedding $e_0 : \mathbb{P} \to \mathbb{P} \times \mathbb{Q}$ defined by $e_0(p) = (p, 1_\mathbb{Q})$ is a complete embedding. Simiarly for $e_1 : \mathbb{Q} \to \mathbb{P} \times \mathbb{Q}$.

Proof. exercise. □

So from a generic of a product poset, we may recover generics from each factor. The following theorem shows that indeed forcing with $\mathbb{P} \times \mathbb{Q}$ does the same thing as forcing with $\mathbb{P}$ and then forcing again with $\mathbb{Q}$.

Theorem 23. Suppose $\mathbb{P}, \mathbb{Q}$ are posets and let G, H be subsets $\mathbb{P}, \mathbb{Q}$ respectively. Then the following are equivalent.

1. $G \times H \subseteq \mathbb{P} \times \mathbb{Q}$ is M-generic
2. G is M-generic and H is $M[G]$-generic
3. H is M-generic and G is $M[H]$-generic

Moreoever, if the above holds, then $M[G \times H] = M[G][H] = M[H][G]$.

I won't prove this, because we'll prove something very similar about a generalization of this construction later. If you're interested in seeing a proof, this is Kunen's chapter VIII, Theorem 1.4

We'll look at examples of product forcing posets. Understandably, they will be product posets (with infinitely many factors!) and contain more information.

Example 24. Let I be an index set and let $\mathbb{P}_i$ be posets for each $i \in I$. Then the *full support product* of the $\mathbb{P}_i$ is the poset $\Pi_{i \in I} \mathbb{P}_i = \{f \mid p : I \to \bigcup_{i \in I} \mathbb{P}_i \wedge p(i) \in \mathbb{P}_i\}$ (we visualize this as the set of all the length-$|I|$ sequences, where the ith place is an element of $\mathbb{P}_i$). The ordering is coordinate-wise: $p \leq q$ iss for all $i \in I$ we have $p(i) \leq_{\mathbb{P}_i} q(i)$.

Example 25. Let I and $\mathbb{P}_i$ be as in the above. Then the *finite support product* of the $\mathbb{P}_i$ is the subposet of the full support product where we only allow conditions p with the property that $p(i) \neq 1_{\mathbb{P}_i}$ finitely often.

Remark. We might think of the conditions p with $p(i) = 1_{\mathbb{P}_i}$ as somehow trivial or containing no useful information. Also note that the choice of support matters a great deal, for reasons that's going to take up too much time. But Joel Hamkins's answer here on Mathoverflow has a nice explanation: https://mathoverflow.net/questions/116564/ill-admit-it-i-dont-understand-the-definition-of-the-easton-product/116579

Here's a type of forcing that we will actually use.

Definition 26. Suppose I is a class of regular cardinals (possibly a proper class). For each $\alpha \in I$ let $\mathbb{P}_\alpha$ be a poset. Then the *Easton support product* of the $\mathbb{P}_\alpha$'s is the collection of functions p with set-sized domain $\mathrm{dom}(p) \subseteq I$ and so that if κ is weakly inaccessible, then $p(\alpha) \neq 1_{\mathbb{P}_\alpha} \neq$ for $< \kappa$ many $\alpha < \kappa$.

The conditions are ordered first by extension of their domain and then coordinate-wise. $p \leq q$ iff $\mathrm{dom}(p) \supseteq \mathrm{dom}(q)$ and for all $i \in \mathrm{dom}(q)$ we have $p(i) \leq q(i)$.

Remark. Following the language in the previous remark, Easton support says that below a weakly inaccessible cardinal, $p(\alpha) \neq 1_{\mathbb{P}_\alpha}$ (that is, nontrivial information is conveyed) can only happen boundedly many times. Also, the reason for requiring only that $\mathrm{dom}(p) \subseteq I$ instead of full equality is because later we will want I to be a proper class. But for now let's look at a set-sized I for simplicity.

Definition 27. Let I be a set of regular cardinals, a function $E : I \to Card$ is called an *Easton index function* iff it satisfies the following two conditions:

1. if $\kappa > \lambda$, then $E(\kappa) \geq E(\lambda)$
2. $cf(E(\kappa)) > \kappa$

Remark. This is meant to be a bookkeeping function. (by the way "bookkeeping/bookkeeper" is the only word in the English language with three consecutive sets of double letters without hyphens, while "subbookkeeper" is the only word found in an English language dictionary with four pairs of double letters in a row.) E is supposed to describe the behavior of the function $\kappa \mapsto 2^\kappa$ in our target extension. Conditions 1 and 2 are ZFC-provable restrictions on the continuum function. The restriction on regular cardinals is because powersets of singular cardinals are very wacky.

Definition 28. Let E be an Easton index function with domain I. Then the *Easton poset* $\mathbb{P}(E) = \Pi_{\alpha\in I} \operatorname{Add}(\alpha, E(\alpha))$ is the product with Easton support.

Proposition 29. Let E be an Easton index function where $\operatorname{dom}(E) = I \subseteq \lambda^+$, where λ is a regular cardinal so that $2^{<\lambda} = \lambda$. Then $\mathbb{P}$. Then $\mathbb{P}(E)$ has the λ^+-cc.

Remark. As we will see later, the requirement that $2^{<\lambda} = \lambda$ is satisfied by all cardinals, if we assume GCH.

Proof. This proof is going to be very similar to the proof that $\operatorname{Add}(\kappa, \lambda)$ has $(2^{<\kappa})^+$-cc.

Let $A = \{p_i \mid i < \lambda^+\}$ be a subset of $\mathbb{P}(E)$, where the p_i's are distinct. We shall prove that there must be some i, j such that p_i, p_j are compatible, making A not an antichain. But first notice we can assume without loss of generality that each p_i has domain I (this is because if p_i is not defined at some $\alpha \in I$, we may consider p_i as mapping α to 1. Doing this doesn't affect compatibility (exercise)).

For each $i < \lambda^+$, let $D_i = \bigcup\{\{\alpha\} \times \operatorname{dom} p_i(\alpha)) \mid \alpha \in I\}$.

Remark: recall that p_i is a function mapping α to a condition $q \in \operatorname{Add}(\alpha, E(\alpha))$. So each D_i tracks the domains of all such q's and keeps them disjoint (this is what the $\{\alpha\} \times \ldots$ does).

Claim: $|D_i| < \lambda$.

proof of claim: If λ is not weakly inaccessible (so regular but not limit), then there are only $< \lambda$ many regular cardinals below λ (for example, how many regular cardinals are below $\aleph_{\omega+1}$ and $\aleph_{\omega+2}$?). On the other hand, if λ is weakly inaccessible (i.e., regular and limit), then the Easton support condition ensures that there can only be boundedly many i's where $p_i(\alpha) \neq 1$ (hence $\operatorname{dom} p_i(\alpha) \neq \emptyset$, recall that the 1 in $\operatorname{Add}(\alpha, E(\alpha))$ is the empty function). That is, $\{\alpha\} \times \operatorname{dom} p_i(\alpha) \neq \emptyset$ can only happen $< \lambda$ many times, so D_i can only have $< \lambda$ many members that are not the empty set. This finishes the proof of claim.

So now we look at $D = \{D_i \mid i < \lambda^+\}$. This is a size λ^+ collection of sets, each of which has size $< \lambda$. Also, $2^{<\lambda} = \lambda$ and the fact that λ is regular implies that $\lambda^{<\lambda} = \lambda < \lambda^+$. The blue parts are conditions of the delta-system lemma. So we may find $J \subseteq D$ a delta system with root R, that is, $D_i \cap D_j = R$ for $i \neq j \in J$.

Now to make $A = \{p_i \mid i < \lambda^+\}$ an antichain, we will need to ensure that for all $(\alpha, s) \in R$, we have that $p_i(\alpha)(s) \neq p_j(\alpha)(s)$ whenever $i \neq j$. For this to happen, we will need to find enough 0-1 sequences of length $|R|$ to tease them apart. But notice that the total number of 0-1 sequences of length $|R|$ is $2^{|R|} \leq 2^{<\lambda} = \lambda < \lambda^+ = |J|$. So by the pigeonhole principle, there must be some $i \neq j \in J$ such that $p_i(\alpha)(s) = p_j(\alpha)(s)$ for all $(\alpha, s) \in R$. But then $q = p_i \cup p_j$ is a common extension. This completes the proof. □

There is a useful representation of $\mathbb{P}(E)$ that allows us to show that $\mathbb{P}(E)$'s preservation properties.

Definition 30. Let E be an Easton index function and λ a cardinal. Then set $E^+_\lambda = E \upharpoonright \{\alpha \mid \alpha > \lambda\}$ and $E^-_\lambda = E \upharpoonright \{\alpha \mid \alpha \le \lambda\}$. Note that this definition applies whether $\operatorname{dom} E$ is a set or not.

Lemma 31. $\mathbb{P}(E) \cong \mathbb{P}(E^-_\lambda) \times \mathbb{P}(E^+_\lambda)$ for any λ.

Proof. exercise. □

Proposition 32. Assume GCH (so that $2^{<\lambda} = \lambda$ for all λ). Let E be an Easton index function. Then forcing with $\mathbb{P}(E)$ preserves regular cardinals (and hence cofinalities and cardinalities).

Proof. Suppose for contradiction that θ is regular in M but not in $M[G]$, where $G \subseteq \mathbb{P}(E)$ is M-generic. Let $\lambda = \operatorname{cof}^{M[G]}(\theta)$ and let $f : \lambda \to \theta$ in $M[G]$ witness that. Recall that cofinality number is a regular cardinal, and regular cardinals are downward absolute. So λ is regular in M as well.

Now we split $\mathbb{P}(E)$ into $\mathbb{P}(E^-_\lambda) \times \mathbb{P}(E^+_\lambda)$ and correspondingly split G into $G^- \times G^+$. Then by our previous theorem about product forcing, we have $M[G] = M[G^-][G^+] = M[G^+][G^-]$.

Claim: $\mathbb{P}(E^+_\lambda)$ is λ^+-closed in M.
proof of claim: if $p_0 \ge p_1 \ge p_2 \ge \ldots$ is a λ-length descending sequence of conditions in $\mathbb{P}(E^+_\lambda)$, then $q = \bigcup_{i<\lambda} p_i$ is also a function with domain $\{\alpha \mid \alpha > \lambda\}$, where $q(\alpha) \le p_i(\alpha)$ for all α, i. This finishes the proof of claim.

So forcing with $\mathbb{P}(E^+_\lambda)$ preserves cardinalities $\le \lambda^+$ and adds no new λ-sequences. In particular, since $M \vDash 2^<\lambda = \lambda$, we have $M[G^+] \vDash 2^<\lambda = \lambda$, and that $\mathbb{P}(E^-_\lambda)^{M[G^+]} = \mathbb{P}(E^-_\lambda)^M$ (this is because conditions in $\mathbb{P}(E^-_\lambda)$ are λ-sequences and G^+ doesn't add any such sequence). So we may apply Proposition 29 in $M[G^+]$ and see that $\mathbb{P}(E^-_\lambda)$ has the λ^+-cc in $M[G^+]$.

But then recall forcing with a λ^+-cc poset implies that $(M[G^+], M[G^+][G^-])$ have the λ^+-cover property. That is, our cofinal function in $M[G^+][G^-]$ can be covered by $F : \lambda \to \mathcal{P}(\theta)$ in $M[G^+]$. This F is a λ-sequence, but G^+ doesn't add new λ-sequences, so F is in the original model M as well. Consider $\bigcup_{i<\lambda} F(i)$, this is a cofinal subset of θ, but has cardinality only $\lambda < \theta$. This contradicts our assumption that θ is regular in M. □

The power of product forcing and Easton posets is the following theorem, which tells us that the only constraint ZFC puts on the continuum function is 1 and 2 in Definition 27. Other than that, the continuum function can behave as wild as you'd like.

Theorem 33 (Easton)**.** Assume GCH. Let E be an Easton index function and suppose $G \subseteq \mathbb{P}(E)$ is M-generic. Then in $M[G]$ we have: for all $\kappa \in \operatorname{dom}(E)$, $2^\kappa = E(\kappa)$.

Proof. From the previous proposition, $\mathbb{P}(E)$ preserves cardinalities. Now fix $\kappa \in \operatorname{dom}(E)$. $2^\kappa \ge E(\kappa)$ is just a general fact we proved about $\operatorname{Add}(\kappa, E(\kappa))$. To see that $2^\kappa \le E(\kappa)$, split $\mathbb{P}(E)$ into $\mathbb{P}(E^-_\kappa) \times \mathbb{P}(E^+_\kappa)$.

Claim: $|\mathbb{P}(E^-_\kappa)| = E(\kappa)$
proof of claim: $E(\kappa) \le |\Pi_{\lambda \le \kappa} \operatorname{Add}(\lambda, E(\lambda))|$ is exercise. Conversely, We notice that for each λ, $|\operatorname{Add}(\lambda, E(\lambda))| = E(\lambda)^{<\lambda}$, which, under GCH, is just $E(\lambda)$, because $E(\lambda)$ has larger cofinality than λ.
So $|\Pi_{\lambda \le \kappa} \operatorname{Add}(\lambda, E(\lambda))|$ is $\Pi_{\lambda \le \kappa} E(\lambda) \le E(\kappa)^\kappa \le_{\mathsf{GCH}} E(\kappa)$. This finishes the proof of claim.

Recall that $\mathbb{P}(E^-_\kappa)$ has the κ^+-cc (proposition 29). So there are $(|\mathbb{P}(E^-_\kappa)|^{<\kappa^+})^\kappa$-many nice names for subsets of κ. By GCH, this is $(E(\kappa)^{<\kappa^+})^\kappa = (E(\kappa)^\kappa)^\kappa = E(\kappa)^\kappa = E(\kappa)$. The last equality holds by GCH and the fact that $\operatorname{cof}(E(\kappa)) > \kappa$. Moreover, as we've seen, forcing with $\mathbb{P}(E^+_\kappa)$ doesn't add κ-sequences, so in $M[G]$ we have $2^\kappa \le E(\kappa)$. □

So we have some pretty liberal control of the continuum function in a generic extension. But there are two worries:
1. So far we've been assuming all the posets are in the ground model. This doesn't necessarily have to be the case. It's possible that we'll need a poset that's not in the ground model but is forced to be in the generic extension.
2. To force over $M \vDash GCH$ and get $M[G] \vDash CCA$, we need to manipulate the continuum function Ord-many times (because we need to code every set of ordinals). But we've observed that this cannot be done when the forcing notion is a set.

In worry 1, the poset might not be in the ground model, but since it will be forced to exist in the extension, this means it has a name in the ground model. Say we've forced with $G \subseteq \mathbb{P}$ to obtain $M[G]$, and we need to force with $\mathbb{Q} \in M[G]$ to obtain some $M[G]H$. In the ground model, we can look at names for $\mathbb{Q}$.

Definition 34. Let $\mathbb{P}$ be a poset and $\dot{\mathbb{Q}}$ be a name os that $1_{\mathbb{P}} \Vdash$ "$\dot{\mathbb{Q}}$ is a poset". Then the *two-step iteration* $P * \dot{Q}$ has domain consisting of pairs $(p, \dot{q})$ with $p \in \mathbb{P}$ and $\dot{q} \in \mathrm{dom}(\mathbb{Q})$ so that $p \Vdash \dot{q} \in \dot{\mathbb{Q}}$. The ordering is defined as follows: $(p_0, \dot{q}_0) \leq (p_1, \dot{q}_1)$ iff $p_0 \leq p_1 \wedge p_0 \Vdash \dot{q}_0 \leq \dot{q}_1$.

Fact 35.

1. $\mathbb{P} * \dot{\mathbb{Q}}$ is a poset
2. if $\mathbb{Q} \in M$, then $\mathbb{P} * \check{\mathbb{Q}}$ is isomorphic to $\mathbb{P} \times \mathbb{Q}$
3. the embedding $e : \mathbb{P} \to \mathbb{P} * \dot{\mathbb{Q}}$ defined by $e(p) = (p, 1_{\mathbb{Q}})$ is a complete embedding. Note: we may assume $1_{\mathbb{Q}} = \emptyset \in M$ here to avoid technicalities about what to designate as $1_{\mathbb{Q}}$.

Definition 36. Suppose $G \subseteq \mathbb{P}$ is M-generic, and let $H \subseteq \dot{Q}_G$. Then $G * H = \{(p, \dot{q}) \in P * \dot{Q} \mid p \in G \wedge \dot{q}_G \in H\}$.

Theorem 37. suppose $K \subseteq \mathbb{P} * \mathbb{Q}$ is M-generic. Let $G = e^{-1}(K) = \{p \in \mathbb{P} \mid (p, 1) \in K\}$ and let $H = \{\dot{q}_G \mid \dot{q} \in \mathrm{dom}(\dot{\mathbb{Q}}) \wedge \exists p \ (p, \dot{q}) \in K\}$. Then $G \subseteq \mathbb{P}$ is M-generic, $H \subseteq \dot{\mathbb{Q}}_G$ is $M[G]$-generic, and $M[K] = M[G][H]$.

Proof. Recall that e defined above is a complete embedding, and so by proposition **??**, G is M-generic. So we now show that H is $M[G]$-generic.

$1_{\mathbb{Q}} \in H$: this is because $(1_{\mathbb{P}}, \dot{1}_{\mathbb{Q}}) \in K$ and by definition of H.

Downward directed: let $q, q' \in H$. Then there are $p, p' \in G$ such that $(p, \dot{q}), (p', \dot{q}') \in K$ (where $q = \dot{q}_G$, $q' = \dot{q}'_G$). Because K is a filter, we may find some $(p'', \dot{q}'') \in K$ that's below both of these conditions. So by definition: $p'' \Vdash \dot{q}'' \leq \dot{q}, \dot{q}'$. Since $p'' \in G$, we have: $q'' = \dot{q}''_G \leq q', q$. But then by definition $q'' \in H$.

Upward closure: let $q = \dot{q}_G \in H$ and $q' \geq q \in \mathbb{Q}$. We want to show that $q' \in H$. By definition of H, there is some $p \in G$ such that $(p, \dot{q}) \in K$. Now since $q' \geq q$ (this is a truth in $M[G]$), then by the truth lemma that there is some $p' \in G$ such that $p' \Vdash \dot{q}' \geq \dot{q}$. By downward directedness of K, there is in K some $(p'', \dot{q}'') \leq (p', 1), (p, \dot{q})$. By definition, $p'' \Vdash \dot{q}'' \leq \dot{q}$. But also $p'' \leq p'$, so $p'' \Vdash \dot{q} \leq \dot{q}'$. So we get that $(p'', \dot{q}'') \leq (p', \dot{q}')$. So by the upward closure of K, it follows that $(p', \dot{q}') \in K$. This means that $q' \in H$.

genericity: take $D = D_G \subseteq \mathbb{Q}$ that is dense in $M[G]$. Take $p_0 \in G$ so that $p_0 \Vdash$ "$\dot{D}$ is dense". Let $D' = \{(p, q) \in \mathbb{P} * \dot{\mathbb{Q}} \mid p \leq p_0 \wedge p \Vdash q \in \dot{D}\}$. Claim: D' is dense below $(p_0, 1)$.
proof of claim: let $(p, q) \leq (p_0, 1)$. We want to find $(p_1, q_1) \in D'$ such that $(p_1, q_1) \leq (p, q)$. By definition, $(p, q) \leq (p_0, 1)$ means that $p \leq p_0$ and $p \Vdash q \leq 1$. But note that $p \Vdash$ "$\dot{D}$ is dense". Then $p \Vdash \exists x \leq q \ (x \in D)$. Let q_1 be such an x (i.e., $p \Vdash q_1 \leq q \wedge q_1 \in \dot{D}$, this is possible by the maximality principle/fullness). Hence (p, q_1) is one such (p_1, q_1) as desired. This finishes the proof of claim.

So there is $(p, \dot{q}) \in K \cap D'$, But then $q = \dot{q}_G \in D$ and $q \in H$. So $H \cap D \neq \emptyset$. Hence H is $M[G]$-generic.

$K = G * H$: $\subseteq$: let $(p, \dot{q}) \in K$, then $p \in G$ and $\dot{q}_G \in H$, So $(p, \dot{q}) \in G * H$. $\supseteq$: let $(p, \dot{q}) \in G * H$. Then $p \in G$ and $\dot{q}_G \in H$. Therefore, $(p, 1) \in K$ and $(p', \dot{q}) \in K$ for some p'. But then there is $(p'', \dot{q}') \leq (p, 1), (p', \dot{q})$ that is in K. So then $p'' \leq p$ and $p'' \Vdash \dot{q}' \leq \dot{q}$. So $(p, \dot{q}) \in K$.

$M[K] = M[G][H]$: note that $K = G * H \in M[G][H]$, so $M[K] \subseteq M[G][H]$ (this is due to the minimality of the generic extension). For the other direction, $G, H \in M[K]$, so $M[G][H] \subseteq M[K]$. $\square$

Remark. two remarks: 1. unlike product forcing, $M[G][H] \neq M[H][G]$. In fact, the latter doesn't even make sense, because $\dot{\mathbb{Q}} * \mathbb{P}$ is undefined. 2. The converse to the above theorem is also true: if $G \subseteq \mathbb{P}$ is M-generic and $H \subseteq \dot{\mathbb{Q}}_G$ is $M[G]$-generic. Letting $K = H * G$, we have: $K \subseteq \mathbb{P} * \dot{\mathbb{Q}}$ is M-generic.

So we know how to force in the ground model with "fictional" objects in the generic extension. I will handwave and state without proof that this can be generalized into transfinite iterations.

Definition 38. Let α be an ordinal. Then the α-stage iterated forcing is a pair of sequences $\langle \mathbb{P}_\xi : \xi \leq \alpha \rangle$ and $\langle \dot{\mathbb{Q}}_\xi : \xi < \alpha \rangle$ so that the following hold:

1. each $\mathbb{P}_\xi$ is a forcing poset
2. each $\dot{\mathbb{Q}}_\xi$ is a $\mathbb{P}_\xi$-name for a forcing poset
3. each $p \in \mathbb{P}_\xi$ is a sequence of the form $\langle \dot{q}_i : i < \xi \rangle$ wgere each $\dot{q}_i \in \text{dom}(\dot{Q}_i)$
4. if $\xi < \eta$ and $p \in \mathbb{P}_\eta$, then $p \restriction \xi \in \mathbb{P}_\xi$
5. if $\xi < \eta$ and $p \in \mathbb{P}_\xi$, and p' is an η-length sequence so that $p' \restriction \xi = p$ and $p'(i) = q_{\dot{\mathbb{Q}}_i}$ for all $i \geq \xi$, then $p' \in \mathbb{P}_\eta$. Let $e^\eta_\xi : \mathbb{P}_\xi \to \mathbb{P}_\eta$ denote the embedding defined by $p \mapsto p'/$
6. $1_{\mathbb{P}_\xi}$ is the sequence $\langle 1_{\dot{\mathbb{Q}}_i} : i < \xi \rangle$
7. for $p, p' \in \mathbb{P}_\xi$, we have $p \leq p'$ iff $p \restriction i \Vdash_{\mathbb{P}_i} p(i) \leq p'(i)$ for all $i < \xi$
8. if $\xi + 1 \leq \alpha$, then $\mathbb{P}_\xi$ is the set of all $p \frown \dot{q}$ such that $p \in \mathbb{P}_\xi$ and $\dot{q} \in \text{dom}(\dot{\mathbb{Q}}_\xi)$ and $p \Vdash_{\mathbb{P}_\xi} \dot{q} \in \dot{\mathbb{Q}}_\xi$.

Remark. what's a good intuition of this? I'll have to admit that I don't have a very good one. But roughly one might think of it this way: you force with $\mathbb{Q}_0$, then $\mathbb{Q}_1$, then $\mathbb{Q}_2$,..., then $\mathbb{Q}_i$,... and so on, for $i < \alpha$. The $\mathbb{P}_\xi$'s represent the iteration of the first ξ many of those, so that $\mathbb{P}_\alpha$ is the total iteration.

So this addresses worry 1. To address worry 2, we use class-sized forcings. We will blackbox class forcing, but I want to remark that not all facts about set-sized forcings generalize to class-sized forcings. But we will restrict our attention to a type of class forcing that does have nice generalizations.

Fact 39. Let $\mathbb{P}$ be an Ord-length iteration of set-sized posets, where the support of each $p \in \mathbb{P}$ is a set. (this is to ensure that each p is a set, so that $\mathbb{P}$ is not a proper class of proper classes). Then $\mathbb{P}$ admits definable forcing relations which satisfy the truth lemma. Moreoever, forcing with $\mathbb{P}$ preserves ZFC.

Generalizing Easton's theorem from last time in a straightforward manner, we have:

Theorem 40 (Easton)**.** Assume GCH. Let E be an Easton index function whose domain is the class of all regular cardinals. Then if $G \subseteq \mathbb{P}(E)$ is M-generic, then we have that $M[G] \vDash 2^\kappa = E(\kappa)$ for all regular κ.

Finally, as promised, we show how to force CCA, and hence GA.

Recall the notion of coding into the continuum pattern:

Definition 3. Recall that if x is a set, then there is some ordinal δ_x and relation $E_x \subseteq \delta \times \delta$ such that $(tc(\{x\}), \in) \cong (\delta_x, E_x)$ Let α, λ be ordinals. Let $g : ON \to ON \times ON$ be the inverse of Gödel's pairing function. Define the set $c(\alpha, \lambda) \subseteq \lambda$ as follows: for all $i \in \lambda$, $i \in c(\alpha, \lambda) \iff 2^{\aleph_{\alpha+i+1}} = \aleph_{\alpha+i+2}$. We say that x *is coded into the continuum pattern at* α *with length* λ iff $g''c(\alpha, \lambda) = E_x$. That is, $\in\restriction tc(\{x\}) \cong g''c(\alpha, \lambda)$.

We say that x *is coded into the continuum pattern* when there is some α, λ such that x is coded into the continuum pattern at α with length λ.

Theorem 41. There is a class forcing $\mathbb{P}$ such that it forces every set (that is, every set in the extension) to be coded into the continuum pattern.

proof.

For simplicity, assume GCH in the ground model (it suffices to assume $V = L$ for example).

Definition 42 (Hamkins)**.** Given posets A and B, we define the *lottery sum* of A and B to be the poset $A \oplus B = \{((A, a), (B, b)) \mid a \in A \wedge b \in B\} \cup \{1_{A\oplus B}\}$, where $1_{A\oplus B}$ is some distinguish element. The ordering is defined to be $(X, x) \leq (Y, y)$ iff $X = Y$ and $x \leq_X y$ and $(X, x) \leq 1_{A\oplus B}$ for all X, x.

Remark. The lottery component of this definition is illustrated by considering some generic $G \subseteq A \oplus B$. Since everything in a filter is compatible with everything else, we conclude that if $(A, x) \in G$, then it's not the case that $(B, y) \in G$. So figuratively, taking a generic is like a lottery of which poset we want to force with.

Definition 43. Let $\mathbb{P} = \mathbb{P}_{\text{Ord}}$ be the full-support iteration $\langle \mathbb{P}_\xi : \xi \in \text{Ord}\rangle, \langle \dot{\mathbb{Q}}_\xi : \xi \in \text{Ord}\rangle$ defined so that $\dot{Q}_\xi$ is a $\mathbb{P}_\xi$-name for the lottery sum of some trivial forcing (any forcing that doesn't add new sets) and $\text{Add}(\aleph_{\xi+1}, \aleph_{\xi+3})$.

Remark. To see what this forcing does: at stage ξ of the iteration, we generically choose whether to do nothing (so we preserve GCH), or to violate GCH at $\aleph_{\xi+1}$. Full support here means that at each limit stage η, the conditions $p \in \mathbb{P}_\eta$ can be non-1 for arbitrarily many $i < \eta$.

Let $G \subseteq \mathbb{P}$ be M-generic. Then $M[G]$ satisfies ZFC. We want to show that every set in $M[G]$ is coded into the continuum pattern. Recall that since transitive ZFC models are determined by their ordinals, it suffices to show that every set of ordinals is coded into the continuum pattern. Just to be clear, we want to show: for every set of ordinals $a \in M[G]$, such that $a \subseteq \gamma$ for some γ, there is $\alpha \in \text{Ord}$ such that for $i < \gamma$ we have that $i \in a \Leftrightarrow 2^{\aleph_{\alpha+i+1}} = 2^{\aleph_{\alpha+i+2}}$.

Claim 44. $\mathbb{P}$ doesn collapse cardinals.

Proof. this is skipped, because we didn't get into the specific properties of class forcing. □

Claim 45. For every $x \in M[G]$, there is $\xi \in \text{Ord}$ such that $x \in M[G_\xi]$, where $G_\xi \subseteq \mathbb{P}_\xi$ is the restriction of G to $\mathbb{P}_\xi$

proof sketch. First, notice that the $\mathbb{Q}_\xi$'s will be more and more closed as ξ-increases. (recall: $\text{Add}(\kappa, \lambda)$ is κ-closed.) So if $x \in M[G]$ and $|x| < \aleph_{\delta+1}$, then x cannot be added later than the δ's stage. □

Fix $a \in M[G]$ a set of ordinals with $a \subseteq \gamma$. Fix ξ such that $a \in M[G_\xi]$.

Claim 46. densely many conditions in the tail forcing above ξ will force a to be coded into the continuum pattern.

Proof. Take a condition p in the tail forcing. Let $\alpha = \sup\{i \mid p(i) \neq 1\}$. Then $\alpha \in \mathrm{Ord}$ (this is because our support is taken to be set-sized, so such i's form a set). Notice that we can extend p to p' by leaving things below α unchanged, and for each $i < \gamma$, we can extend $p(\alpha + i)$ so $p'(\alpha + i)$ is in the trivial part if $i \notin a$ and $p'(\alpha + i)$ is in the Add part if $i \in a$. Then, any generic containing p' will code a into the continuum pattern starting at α. □

So by density, if G^ξ is the restriction of G to the tail forcing, then G^ξ will force a to be coded into the continuum pattern. Taking stock, what we've shown is that for any arbitrary set of ordinals $a \in M[G]$, a is coded into the continuum pattern. Hence, every set is coded into the continuum pattern. □Theorem 41

Corollary 47. $\mathrm{Con}(\mathsf{ZFC}) \rightarrow \mathrm{Con}(\mathsf{ZFC} + GA + \neg\mathsf{GCH})$.

Coding sets into the continuum pattern is a versatile tool. For example, every set is consistently hereditarily ordinal definable.

Definition 48. A set X is *ordinal definable* iff there is some formula $\varphi(x, \vec{\alpha})$ with one free variable x and some (possible none) ordinal parameters $\vec{\alpha}$, such that for any a, $a \in X \Leftrightarrow \varphi(a, \vec{\alpha})$. A set X is *hereditarily ordinal definable* iff every set in the transitive closure of $\{X\}$ is ordinal definable.

The class of all ordinal definable sets is denoted OD and the class of all hereditarily ordinal definable sets is denoted HOD.

Fact 49. HOD is transitive, and HOD $\models$ ZFC.

Fact 50. HOD is the largest transitive model of ZFC for which there exists a definable one-to-one correspondence with the class of all ordinals.

Proposition 51. Let X be a set in V. Then there is a generic extension $V[G]$ such that $X \in (\mathrm{HOD})^{V[G]}$.

proof sketch. Code the transitive closure of $\{X\}$ into the continuum pattern (so there is a set of ordinal E_X such that $\Gamma\text{"}E_X$ is isomorphic to $\in\restriction tc(\{X\})$). So in the extension, there is some θ such that in $V[G]$, $2^{\aleph_{\theta+i+1}} = \aleph_{\theta+i+2}$ iff $i \in E_X$. So in the extension $V[G]$, we may define E_X as $\{i \mid 2^{\aleph_{\theta+i+1}} = \aleph_{\theta+i+2}\}$. So this set of ordinals is in HOD. By our proposition two lectures ago, X is in HOD. □

A small variation of the technique above gives us the *universal definition phenomenon* (Hamkins) in set theory:

Proposition 52 (the simplified case of universal definition of a real)**.** there is a definition $\varphi(x)$ such that for any real $a \subseteq \omega$, there is a forcing extension such that $\varphi(x)$ defines a. That is, there is a unique x such that $\varphi(x)$ and $x = a$. More snappily, any particular real number r can become definable in a forcing extension of the universe.

proof sketch. Given $r \subseteq \omega$. Use Easton's theorem to make $2^{\aleph_{n+1}} = \aleph_{n+2}$ or $2^{\aleph_{n+1}} \neq \aleph_{n+2}$, depending on whether $n \in r$. □