MEASURE-PRESERVING TRANSFORMATIONS WITH ROOTS ARE COMPLETE ANALYTIC

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Abstract. For an integer $n \geq 2$, let

$$\mathcal{R}_n = \{T \in \mathrm{Aut}([0,1], \lambda) : (\exists S \in \mathrm{Aut}([0,1], \lambda))\ S^n = T\}.$$

We prove that $\mathcal{R}_n$ is $\mathbf{\Sigma}^1_1$-complete for every $n \geq 2$. More generally, for every nonempty $A \subseteq \{2, 3, \ldots\}$, the set of transformations admitting an nth root for some $n \in A$ is $\mathbf{\Sigma}^1_1$-complete. The case $n = 2$ settles the descriptive-complexity conjecture in Problem 43 of the Tianyuan Workshop problem list. The proof combines the Foreman–Rudolph–Weiss reduction from ill-founded trees to transformations conjugate to their inverses with a consequence of their description of the centralizer: the transformations in the reducing family admit no mth root for any $m \geq 2$.

1. Introduction

Let (X, μ) be a standard atomless probability space, and let

$$G = \mathrm{Aut}(X, \mu)$$

be the group of invertible measure-preserving transformations modulo equality almost everywhere, equipped with the weak topology. Throughout, $\cong$ denotes measure-theoretic conjugacy. For $n \geq 2$, put

$$\mathcal{R}_n = \{T \in G : (\exists S \in G)\ S^n = T\}.$$

The map $S \mapsto S^n$ is continuous, and hence $\mathcal{R}_n$ is analytic.

Halmos [4, p. 97] asked for a characterization of $\mathcal{R}_2$ (which he originally studied in [3]). At the 2025 Tianyuan Workshop on Computability Theory and Descriptive Set Theory, Thornton conjectured that $\mathcal{R}_2$ is complete analytic; see Problem 43 of [1].

We prove the following stronger statement.

Theorem 1. *For every integer $n \geq 2$, the set $\mathcal{R}_n$ is $\mathbf{\Sigma}^1_1$-complete. The continuous reduction witnessing hardness takes values among transformations having exactly n ergodic components, each of measure $1/n$.*

Foreman, Rudolph, and Weiss construct in [2] a continuous family of ergodic transformations satisfying

$$\tau \in \mathsf{IF} \iff R_\tau \cong R_\tau^{-1},$$

where IF is the complete analytic set of ill-founded trees. We first deduce from their analysis of graph joinings that no R_τ has an mth root for any

$m \geq 2$. We then apply the finite direct-sum construction

$$D_n(R) = \underbrace{R \oplus \cdots \oplus R}_{n-1 \text{ copies}} \oplus R^{-1}.$$

For the transformations in the Foreman–Rudolph–Weiss family, $D_n(R)$ has an nth root exactly when R is conjugate to R^{-1}.

2. The Foreman–Rudolph–Weiss family

Let Tr be the Polish space of trees on $\mathbb{N}$ of unbounded height used in [2], and let $\mathsf{IF} \subseteq \mathsf{Tr}$ be the set of ill-founded trees. The set IF is $\mathbf{\Sigma}^1_1$-complete [2, Theorem 4].

Let $\tau \mapsto \nu_\tau$ be the continuous map of Theorem 7 of [2], and let K_τ be the support of ν_τ. Thus $(K_\tau, \nu_\tau, \mathrm{sh})$ is ergodic and

$$\tau \in \mathsf{IF} \iff (K_\tau, \nu_\tau, \mathrm{sh}) \cong (K_\tau, \nu_\tau, \mathrm{sh}^{-1}). \tag{1}$$

Foreman, Rudolph, and Weiss observe that the same construction may be carried out by cutting and stacking on the unit interval. In this form it yields a continuous map

$$\tau \longmapsto R_\tau \in G$$

such that R_τ is measure-theoretically isomorphic to $(K_\tau, \nu_\tau, \mathrm{sh})$ for every τ; see [2, Section 2.2 and the theorem following Corollary 9]. We use this realization below. The argument establishing the absence of roots is carried out in the symbolic model and then transferred to R_τ by conjugacy.

For a fixed $\tau \in \mathsf{Tr}$, the symbolic system has an odometer factor

$$\pi_\tau : K_\tau \longrightarrow \mathcal{O}_\tau$$

such that

$$\pi_\tau(\mathrm{sh}\, x) = \pi_\tau(x) + \overline{1} \tag{2}$$

for almost every x [2, Section 5.1 and Lemma 22]. The coordinate moduli of $\mathcal{O}_\tau$ form an infinite sequence $q_0, q_1, \ldots$, with $q_i \geq 2$ for every i. We write $\overline{j} = j\overline{1}$ for $j \in \mathbb{Z}$.

Lemma 2. *Let Q be an automorphism of (K_τ, ν_τ) that commutes with* sh. *Then there is an integer j such that*

$$\pi_\tau(Qx) = \pi_\tau(x) + \overline{j}$$

for almost every x.

Proof. Let $\Delta_Q = (\mathrm{id}, Q)_* \nu_\tau$ be the graph joining of Q. Since $Q\,\mathrm{sh}\,Q^{-1} = \mathrm{sh}$, Corollary 40 of [2] gives an integer j and an even element g of the inverse-limit group $G_\infty(\tau)$ such that $\eta_g \circ (1, \mathrm{sh}^j)$ is supported on the graph of Q. Every joining has first marginal ν_τ, so a joining supported on this graph is necessarily Δ_Q.

By Lemmas 26 and 27 of [2], together with the paragraph following Lemma 27, the graph joining η_g associated with an even g is a self-joining of K_τ whose projection to $\mathcal{O}_\tau \times \mathcal{O}_\tau$ is concentrated on the diagonal. Applying

sh^j in the second coordinate changes this projection to the graph of the translation

$$z \longmapsto z + \overline{j}.$$

The asserted identity follows. $\square$

Proposition 3. *For every $\tau \in \mathsf{Tr}$ and every integer $m \geq 2$, the transformation R_τ has no mth root.*

Proof. Root existence is invariant under measure-theoretic conjugacy, so it is enough to work with the symbolic system. Suppose, toward a contradiction, that

$$Q^m = \mathrm{sh}$$

for some $m \geq 2$. Then Q commutes with sh. By Lemma 2, there is $j \in \mathbb{Z}$ such that

$$\pi_\tau(Qx) = \pi_\tau(x) + \overline{j}$$

for almost every x. Iterating this identity and using (2), we obtain

$$\pi_\tau(x) + \overline{mj} = \pi_\tau(Q^m x) = \pi_\tau(\mathrm{sh}\, x) = \pi_\tau(x) + \overline{1}.$$

Thus

$$(mj - 1)\overline{1} = 0$$

in $\mathcal{O}_\tau$.

The element $\overline{1}$ has infinite order. Indeed, for each r, the quotient determined by the first $r+1$ odometer coordinates is cyclic of order

$$q_0 q_1 \cdots q_r,$$

and the image of $\overline{1}$ has that order. Since these orders are unbounded, $k\overline{1} = 0$ implies $k = 0$ for every $k \in \mathbb{Z}$. Consequently $mj = 1$, which is impossible for $m \geq 2$ and $j \in \mathbb{Z}$. $\square$

We shall use the following consequences of the Foreman–Rudolph–Weiss construction:

$$\begin{aligned} &\tau \longmapsto R_\tau \text{ is continuous, and every } R_\tau \text{ is ergodic;} \\ &\tau \in \mathsf{IF} \Longleftrightarrow R_\tau \cong R_\tau^{-1}; \\ &\text{no } R_\tau \text{ has an } m\text{th root for any } m \geq 2. \end{aligned} \tag{3}$$

3. Finite direct sums

For $n \geq 2$, let

$$X_n = X \times \{0, 1, \dots, n-1\}, \qquad \mu_n = \mu \times u_n,$$

where u_n is the uniform probability measure. The space (X_n, μ_n) is a standard atomless probability space. Fix a measure-space isomorphism between (X_n, μ_n) and (X, μ). For $T_0, \dots, T_{n-1} \in G$, the direct sum is defined on X_n by

$$(T_0 \oplus \cdots \oplus T_{n-1})(x, i) = (T_i x, i),$$

and is regarded, through the fixed isomorphism, as an element of G.

Lemma 4. *For each $n \geq 2$, the map*

$$G^n \longrightarrow G, \qquad (T_0, \dots, T_{n-1}) \longmapsto T_0 \oplus \cdots \oplus T_{n-1},$$

is continuous. Consequently, the map

$$D_n : G \longrightarrow G, \qquad D_n(R) = \underbrace{R \oplus \cdots \oplus R}_{n-1 \text{ copies}} \oplus R^{-1},$$

is continuous.

Proof. It is enough to verify continuity on (X_n, μ_n) before applying the fixed identification with (X, μ), since conjugation by that identification is a homeomorphism of the corresponding automorphism groups. Suppose that $T_i^{(k)} \to T_i$ in the weak topology for every $i < n$. For a measurable set $A \subseteq X_n$, let

$$A_i = \{x \in X : (x, i) \in A\}.$$

Then

$$\mu_n\left(\left(\bigoplus_{i<n} T_i^{(k)}\right) A \,\triangle\, \left(\bigoplus_{i<n} T_i\right) A\right) = \frac{1}{n} \sum_{i<n} \mu\big(T_i^{(k)} A_i \,\triangle\, T_i A_i\big) \longrightarrow 0.$$

This is convergence in the weak topology. The final assertion follows because inversion is continuous in the Polish group G. □

Proposition 5. *Let $R \in G$ be ergodic and suppose that R has no mth root for any $m \geq 2$. For every $n \geq 2$,*

$$D_n(R) \in \mathcal{R}_n \iff R \cong R^{-1}.$$

Moreover, if $R \not\cong R^{-1}$, then $D_n(R)$ has no mth root for any $m \geq 2$.

Proof. Suppose first that $R \cong R^{-1}$. Then $D_n(R)$ is conjugate to the direct sum $R^{\oplus n}$ of n copies of R. On X_n, define

$$C(x, i) = \begin{cases} (x, i+1), & i < n-1, \\ (Rx, 0), & i = n-1. \end{cases}$$

The map C is invertible and measure preserving, and

$$C^n = R^{\oplus n}.$$

It follows that $D_n(R)$ has an nth root.

Conversely, suppose that $R \not\cong R^{-1}$ and that

$$S^m = D_n(R)$$

for some $m \geq 2$. Then S commutes with $D_n(R)$. Let

$$E_i = X \times \{i\}, \qquad i < n.$$

An invariant measurable set meets each E_i in a null or conull set, since the restriction of $D_n(R)$ to E_i is ergodic. Thus $E_0, \dots, E_{n-1}$ are precisely the atoms of the invariant σ-algebra of $D_n(R)$. Commutation with $D_n(R)$ implies that S preserves this σ-algebra, and hence permutes its atoms modulo null sets.

The atom E_{n-1} is the only one on which the induced transformation is isomorphic to R^{-1}. It cannot be carried by S to one of the other atoms, because the corresponding restriction of S would conjugate R^{-1} to R. Hence $S(E_{n-1}) = E_{n-1}$ modulo null sets. After identifying E_{n-1} with X, the restriction of the equation $S^m = D_n(R)$ to this component gives an automorphism Q satisfying

$$Q^m = R^{-1}.$$

Then $(Q^{-1})^m = R$, contrary to the hypothesis on R. Thus $D_n(R)$ has no mth root for any $m \geq 2$ whenever $R \not\cong R^{-1}$. □

4. Proof of the completeness theorem

Proof of Theorem 1. Fix $n \geq 2$. As observed above, $\mathcal{R}_n$ is analytic because it is the range of the continuous map $S \mapsto S^n$ on the Polish space G.

Define

$$\Phi_n : \mathsf{Tr} \longrightarrow G, \qquad \Phi_n(\tau) = D_n(R_\tau).$$

The map Φ_n is continuous by Lemma 4. By Proposition 5 and (3),

$$\tau \in \mathsf{IF} \quad \Longleftrightarrow \quad R_\tau \cong R_\tau^{-1} \quad \Longleftrightarrow \quad \Phi_n(\tau) \in \mathcal{R}_n.$$

Thus Φ_n continuously reduces the complete analytic set IF to $\mathcal{R}_n$. It follows that $\mathcal{R}_n$ is $\mathbf{\Sigma}^1_1$-complete.

Since every R_τ is ergodic, the transformation $\Phi_n(\tau)$ has exactly n ergodic components, namely its coordinate components, and each has measure $1/n$. □

Taking $n = 2$ gives the conjecture from Problem 43.

Corollary 6. *The set*

$$\{T \in \mathrm{Aut}([0,1], \lambda) : (\exists S \in \mathrm{Aut}([0,1], \lambda))\ S^2 = T\}$$

is $\mathbf{\Sigma}^1_1$-complete.

The last assertion of Proposition 5 gives a further consequence.

Corollary 7. *Let A be any nonempty subset of $\{2, 3, \ldots\}$. Then*

$$\mathcal{R}_A = \{T \in G : (\exists n \in A)(\exists S \in G)\ S^n = T\} = \bigcup_{n \in A} \mathcal{R}_n$$

is $\mathbf{\Sigma}^1_1$-complete. In particular, the set of transformations admitting an mth root for at least one $m \geq 2$ is $\mathbf{\Sigma}^1_1$-complete.

Proof. The set $\mathcal{R}_A$ is analytic as a countable union of analytic sets. Choose $n \in A$ and use the map Φ_n from the proof of Theorem 1. If $\tau \in \mathsf{IF}$, then

$\Phi_n(\tau)$ has an nth root. If $\tau \notin \mathsf{IF}$, then $R_\tau \ncong R_\tau^{-1}$, so Proposition 5 shows that $\Phi_n(\tau)$ has no mth root for any $m \geq 2$. Hence

$$\tau \in \mathsf{IF} \quad \Longleftrightarrow \quad \Phi_n(\tau) \in \mathcal{R}_A.$$

$\square$

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