Inaccessibility

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1 Prelims

Definition 1.1. A linear ordering $<$ on a set P is a *well-ordering* if every nonempty subset of P has a $<$-least element. A set x is *transitive* iff $y \in x$ implies $y \subseteq x$. Or equivalently, $z \in y \wedge y \in x \Rightarrow z \in x$

Definition 1.2. An ordinal is a transitive set that's well-ordered by $\in$. When context is clear, we may write $<$ for $\in$.

Fact 1.3.

1. for each ordinal α, $\alpha = \{\beta \mid \beta < \alpha\}$.
2. $0 = \emptyset$ is an ordinal.
3. if α is an ordinal, then $\alpha \cup \{\alpha\}$ is an ordinal; in particular, $\alpha \cup \{\alpha\} = \inf\{\beta \mid \beta > \alpha\}$.
4. if X is a set of ordinal, then $\bigcup X$ is an ordinal; in particular, $\bigcup X = \sup X$.

Example. $2 = \{\emptyset, \{\emptyset\}\} = \{0, 1\}$; $\omega = \{0, 1, 2, 3, ...\}$; $\omega + \omega = \{0, 1, 2, 3, ..., \omega + 1, \omega + 2, \omega + 3, ...\}$

Definition 1.4. Given two sets A, B we write $|A| = |B|$ iff there is a bijection between them; $|A| \leq |B|$ iff there's an injection $f : A \to B$. $|A| < |B|$ iff $|A| \leq |B|$ and $|A| \neq |B|$.

Theorem 1.5. (Cantor) $|X| < |\mathcal{P}(X)|$ for all X.

Definition 1.6.
(i) An ordinal α is called a cardinal if $|\alpha| \neq |\beta|$ for all $\beta < \alpha$. In other words, we identify cardinals with initial ordinals.
(ii) if W is a well-ordered set, then there exists an ordinal α such that $|W| = |\alpha|$. Let $|W|$ denote the least α such that $|W| = |\alpha|$.

Theorem 1.7. (Zermelo)(AC) Every set can be well-ordered.

Theorem 1.8. (Replacement) Every well-oredered set is isomorphic to a (unique) ordinal.

Fact 1.9.
(i) for every α there is a cardinal number greater than α.
(ii) if X is a set of cardinals, then $\sup X$ is a cardinal.

If κ is a cardinal, we let κ^+ denote the least cardinal greater than κ.

Definition 1.10. (Aleph numbers)
$\aleph_0 = \omega$
$\aleph_{\alpha+1} = \aleph_\alpha^+$
$\aleph_\lambda = \sup\{\aleph_\beta \mid \beta < \alpha\}$, for limit ordinal λ.
(A lot of sources write ω_α and $\aleph_\alpha$ interchangeably.)

Definition 1.11. κ is a successor cardinal iff there is some γ such that $\kappa = \lambda^+$. Otherwise κ is a limit cardinal.

Remark. Limit cardinals are kind of large. They cannot be reached by cardinal successors from below.

Definition 1.12. If α, γ are limit ordinals, then a function $f : \gamma \to \alpha$ is *cofinal* if it's order-preserving and its range is unbounded in α, i.e., for each $x \in \alpha$, there is some $y \in \gamma$ such that $x < f(y)$. The *cofinality* of α, $\mathrm{cf}(\alpha)$ is the least γ such that there is a cofinal function $f : \gamma \to \alpha$.

Example. $\mathrm{cf}(\omega + \omega) = \omega$, as witnessed by $f : n \mapsto \omega + n$. $\mathrm{cf}(\aleph_\omega) = \omega$, witnessed by $f : n \mapsto \aleph_n$.

Proposition 1.13. Let κ be an infinite cardinal. Then $\mathrm{cf}(\kappa)$ is the least α such that there are cardinals $\kappa_\xi < \kappa, \xi < \alpha$ such that $\kappa = \sum_{\xi<\alpha} \kappa_\xi = |\bigcup_{\xi<\kappa}(\kappa_\xi \times \{\xi\})|$ (the cardinality of the disjoint union of the κ_ξ's.)

Remark. this means that cofinality is some measure of largeness: it measures how many pieces of sets smaller than κ is needed to put together a set of size κ.

Proof. Let $\lambda = \mathrm{cf}(\kappa)$ and α as in the above. We want to show that $\lambda = \alpha$.

Let $(\gamma_\xi \mid \xi < \lambda)$ be cofinal in κ. This sequence is well defined because there is a function $f : \lambda \to \kappa$ that's cofinal. For each $\xi < \lambda$, $\gamma_\xi < \kappa$ (this is because κ is a cardinal, see definition 1.6.(i)). By definition of cofinality, $\kappa = \bigcup_{\xi<\lambda} \gamma_\xi$. But $\bigcup_{\xi<\lambda} \gamma_\xi = \sum_{\xi<\lambda} |\gamma_\xi|$ (this equality is not entirely trivial. Proof in appendix). Hence $\alpha \leq \lambda$.

Now suppose for contradcition that $\alpha < \lambda$. Let $(\kappa_\xi)_{\xi<\alpha}$ be such that $\kappa = \sum_{\xi<\kappa} \kappa_\xi$. Since λ is the least ordinal to have a cofinal sequence in κ, we know that $(\kappa_\xi)_{\xi<\alpha}$ is not cofinal in α. In other words, this sequence is bounded by some $\gamma < \kappa$. But $\sum_{\xi<\alpha} \kappa_\xi \leq \sum_{\xi<\alpha} |\gamma| = |\alpha| \times |\gamma| < \kappa$. Contradiction. (The last inequality follows from a general fact about cardinal multiplication: $\kappa \times \lambda = \max\{\kappa, \lambda\}$ if at least one of κ, λ is infinite and neither is zero). $\square$

Remark. So regular cardinals are kind of large. They cannot be put together by smaller pieces of smaller sets.

2 Inaccessibility

Definition 2.1. A cardinal κ is a strong limit iff for all $\lambda < \kappa$, $2^\lambda = |\mathcal{P}(\lambda)| < \kappa$.

Remark. so strong limits are also kind of large, they cannot be reached from below by taking power sets. Also, this definition might not make sense without choice. Powersets are not guaranteed to be well-oredered. See *Inaccessible Cardinals without the Axiom of Choice* by Blass, Dimitriou, Löwe

Definition 2.2.
(i) a weakly-inaccessible cardinal is an uncountable regular limit cardinal;
(ii) a strongly-inaccessible cardinal is an uncountable regular strong limit cardinal.

Proposition 2.3. Let κ be weakly inaccessible, then
(i) $\kappa = \aleph_\kappa$
(ii) $A = \{\alpha \in \kappa \mid \alpha = \aleph_\alpha\}$ is closed unbounded in κ.

Proof.
(i) Let $\kappa = \aleph_\lambda$ for some limit ordinal λ. We show that $\kappa = \lambda$.

$$\begin{aligned}
\kappa &= \text{cf}(\kappa) \quad \text{(by regularity)} \\
&= \text{cf}(\aleph_\lambda) \quad \text{(by definition)} \\
&= \text{cf}(\lambda) \quad (\kappa \text{ is a limit cardinal}) \\
&\leq \lambda \quad \text{(implied by definition of cofinality)} \\
&\leq \aleph_\lambda = \kappa \quad \text{(the last inequality can be proven by induction)}
\end{aligned}$$

(ii) Let $(\alpha_i \mid i < \lambda)$, λ limit, be a strictly increasing sequence of elements in A that's not cofinal in κ. To show that A is closed, it suffices to show that the sup of this sequence is in A. Let $\alpha_\lambda = \bigcup_{i<\lambda} \alpha_i$. We want to show that $\alpha_\lambda = \aleph_{\alpha_\lambda}$.

Suppose for contradiction that $\alpha_\lambda < \aleph_{\alpha_\lambda}$. Then $|\alpha_\lambda| = |\aleph_\gamma|$ for some $\gamma < \alpha_\lambda$. But since α_λ is the least upper bound of the α_i's, there must be some α_k in the sequence that's above γ (if not, then γ would be an upper bound for the α_i's, contradicting the minimality of α_λ). But for this α_k, we have $\alpha_\lambda \leq \aleph_{\alpha_k} = \alpha_k < \alpha_\lambda$, a contradiction.

To show that A is unbounded, let $\beta \in \kappa$. We want to find some $\alpha \in A$ that's above β. First we note that κ is an $\aleph$ fixed-point. So $\beta \leq \aleph_\beta < \kappa$ (suppose not, then $\aleph_\beta \geq \aleph_\kappa = \kappa$, which implies that $\beta \geq \kappa$, contradiction). We define the following ω-sequence:

$$\begin{aligned}
\beta_0 &= \beta \\
\beta_{n+1} &= \aleph_{\beta_n} \\
\beta_\omega &= \bigcup_{k<\omega} \beta_k
\end{aligned}$$

By mathematical induction, we can show that for each n, $\beta_n < \kappa$ using the same argument as in the above parenthesis. And since κ is regular, this ω-sequence cannot be unbounded in κ. So $\beta_\omega < \kappa$.

Now we show $\beta_\omega \in A$. That is, $\beta_\omega = \aleph_{\beta_\omega}$. Suppose for contradiction that $\beta_\omega < \aleph_{\beta_\omega}$. Then there is some n such that $\beta_\omega \leq \aleph_{\beta_n} < \aleph_{\beta_{n+5}} = \beta_{n+6} < \beta_\omega$. Contradiction. □

Proposition 2.4. Let κ be strongly inaccessible, then $S = \{\gamma < \kappa \mid \gamma \text{ is strong limit}\}$ is club in κ. (the proof really closely mirrors 2.3)

Proof. Let κ and S be as in the statement. We first show that S is closed.

Let $(\alpha_i \mid i < \lambda)$, λ a limit ordinal, be a strictly increasing sequence of elements of S that is not cofinal in κ. It suffices to show that the sup of this sequence is in κ. Let $\alpha = \sup(\alpha_i \mid i < \lambda)$. Clearly, α is a limit cardinal. Now let $\beta < \alpha$, we want to show that $2^\beta < \alpha$. Since α is the limit of a sequence of strong limit cardinals, there must be some strong limit cardinal α_k in the sequence such that $\alpha_k > \beta$ (otherwise β would be an upper bound for the $\alpha_i' s$). But α_k is a strong limit, so $2^\beta < \alpha_k < \alpha$. This shows that S is closed in κ. Note that this show, more generally, that a limit of strong limit cardinals is also a strong limit cardinal, a fact that we'll use next.

Now we show that S is unbounded. Let $\beta \in \kappa$. It suffices to show that there is some $\alpha \in S$ such that $\beta < \alpha$. To find this α, we define the following ω-sequence of $\beth$ numbers ("Beth"):

$$\beth_0(\beta) = 2^\beta$$
$$\beth_{n+1}(\beta) = 2^{\beth_n(\beta)}$$
$$\beth_\omega(\beta) = \bigcup_{k<\omega} \beth_k = \sup\{2^{\aleph_0}, 2^{2^{\aleph_0}}, 2^{2^{2^{\aleph_0}}}, ...\}$$

That $\beth_\omega(\beta) < \kappa$ follows from κ being a strong limit and has uncountable cofinality. That $\beth_\omega(\beta) \in S$ follows from it being a limit of strong limits (see above). This shows that S is unbounded. □

Corollary 2.5. If κ is the least strongly inaccessible, then the set of all singular strong limits below it is club. This follows from the observation that all strong limits below κ are singular.

Definition 2.6. The von-Neumann universe is defined by recursion:

$$V_0 = \emptyset$$
$$V_{(\alpha+1)} = \mathcal{P}(V_\alpha)$$
$$V_\lambda = \bigcup_{\beta<\lambda} V_\beta \ \lambda \text{ a limit ordinal}$$

Fact. The V_α's are transitive.

Proposition 2.7. Let κ be strongly inaccessible, then $V_\kappa \vDash ZFC$.

The proof of the proposition is broken down into a few lemmas:

Lemma. Extensionality and Foundation hold in any transitive set.

Proof. Let M be transitive.
-Extensionality: suppose $x \neq y$ for some $x, y \in M$. Then axiom of extensionality in V implies that there is some z such that z is in one of x, y but not the other. Since M is transitive, we conclude that z is in M too.
-Foundation: given transitive M, then Foundation in V implies that there is $x \in M$ such that $x \cap M = \emptyset$. Because M is transitive, $x \subseteq M$. Hence $x \cap M \subseteq x$, we conclude that $x = \emptyset$. □

Lemma. If α is a limit ordinal, then Extensionality, Foundation, Infinity, Separation, Pairing, Union, Powerset, and Choice all hold in V_α.

Proof. Infinity: follows from the fact that $\omega \in V_\mu$ for all $\mu > \omega$.

Separation: let $Y \subseteq X \in V_\alpha$. Because $X \in V_\alpha$, we know that there is some $\beta < \alpha$ such that $X \subseteq V_\beta$. So $Y \subseteq X \subseteq V_\beta$; hence $Y \in V_{\beta+1} \subseteq V_\alpha$.

Union: let $A \in V_\alpha$. It follows that $A \subseteq V_\beta$ for some $\beta < \alpha$, and hence $\bigcup A \subseteq V_\gamma$ for some $\gamma < \alpha$. So $\bigcup A \in V_{\gamma+1} \subseteq V_\alpha$.

Pairing and Powerset: Given that α is limit, if $x, y \in V_\beta$ for some $\beta < \alpha$ then $\{x, y\} \subseteq V_\beta$, and so $\{x, y\} \in V_{\beta+1}$. Similarly for Powerset.

Choice: we use the equivalent form: for all X there is a function $f : \mathcal{P}(X \setminus \{\emptyset\}) \to X$ such that $f(Y) \in Y$ for all $Y \in X \setminus \{\emptyset\}$. Given $X \in V_\alpha$, it follows that $X \in V_\beta \subset V_\alpha$, so $\mathcal{P}(X) \in V_{\beta+1}$. A function from $\mathcal{P}(X \setminus \{\emptyset\})$ to X is an element of $\mathcal{P}(\mathcal{P}(X \setminus \{\emptyset\}) \times X)$. Under the Kuratowski definition of ordered pairs, each choice function is an element of $\mathcal{PPP}(\mathcal{P}(X \setminus \{\emptyset\}) \cup X)$. So if such a choice function exists in V, then it also exists in $V_{\beta+20} \subset V_\alpha$, for instance. □

Lemma. If κ is strongly inaccessible and $\beta < \kappa$, then $|V_\beta| < \kappa$.

Proof. This can be proven by induction: the successor case follows from definition. For the limit case, suppose $|V_\alpha| < \kappa$ for all $\alpha < \beta$. Then $|V_\beta| = |\bigcup_{\alpha<\beta} V_\alpha| \leq \beta \times \kappa = \kappa$. Now suppose for contradiction that $|V_\beta| = \kappa$. But note that $|V_\beta| = \sup|V_\alpha|_{\alpha<\beta}$ (follows from $\kappa \times \kappa = \kappa$). Then $f : \beta \to \kappa$ defined by $f(\alpha) = |V_\alpha|$ is cofinal in κ, which is a contradiction. □

Lemma. If κ is strongly inaccessible, then V_κ satisfies Replacement.

Proof. Let $A \in V_\kappa$ and $F : A \to V_\kappa$ be a definable function. Since κ is a limit, there is $\beta < \kappa$ such that $A \in V_\beta$. Since the V_α's are transitive, $A \subseteq V_\beta$ and $|A| \leq |V_\beta| < \kappa$, by lemma above. So $\{\text{rank}(x) \mid x \in F``A\}$ will be a set of ordinals below $\kappa$, but $\kappa$ is regular, so this set must bounded by some ordinal below $\kappa$. This means that $F``A$ will have rank below κ, and so it's a member of V_κ. This completes the proof of 2.7 □

Corollary 2.8. If ZFC is consistent, then ZFC cannot prove the existence of a strongly inaccessible cardinal. (This can also be proven without appeal to the second incompleteness theorem, by noting that strong inaccessibility is absolute between V and V_κ)

Proposition 2.9. Let ZFC^2 denote the axioms of ZFC with Replacement (and Separation) replaced by a single axiom with second order quantifier. Then $V_\kappa \vDash ZFC^2$ iff κ is strongly inaccessible.

Proof.
($\Leftarrow$) same as above

($\Rightarrow$) V_κ satisfies Infinity, so κ is uncountable.

κ is regular: suppose not, then there is some function $f : \alpha \to \kappa$ with cofinal image, $\alpha < \kappa$. But this function will be a subset of V_κ. But then by second order Replacement, $f``\alpha \in V_\kappa$. By Union, $\sup(f``\alpha) = \kappa \in V_\kappa$, contradiction.

κ is strong limit: suppose not, then there is some $\lambda < \kappa$ such that $2^\lambda \geq \kappa$. $\mathcal{P}(\lambda) \in V_\kappa$ since V_κ satisfies Powerset. Then there is a surjection $H : \mathcal{P}(\lambda) \to \kappa$; by second order Replacement again, we have $H``\mathcal{P}(\lambda) = \kappa \in V_\kappa$. □

Corollary 2.10. There is an strongly inaccessible cardinal iff there is a model of ZFC^2 of the form $(M, E, \mathcal{P}(M))$.

Fact.(Zermelo's Quasi-Categoricity) If the two structures $(M, E, \mathcal{P}(M))$, $(M', E', \mathcal{P}(M)')$ are models of ZFC^2, then one of them will be isomorphic to a substructure of the other.
Fact. There is a model of ZFC^2 of the form $(M, E, \mathcal{P}(M))$ such that if it satisfies CH, then all models of ZFC^2 in this form will satisfy CH. (Because CH is a statement about $V_{\omega+3}$). Kreisel used this fact to argue that CH has a definite truth value. See *Informal Rigour and Completeness Proofs*.

Weak inaccessibility can also be generalized.

Definition 2.11.
(i) κ is 0-weakly inacecssible iff κ is regular;
(ii) κ is $\alpha + 1$-weakly inaccessible iff κ is a regular limit of α-weakly inaccessible cardinals;
(iii) κ is δ-weakly inacccessible iff κ is α-weakly inaccessible for every $\alpha < \delta$ ($\delta > 0$ is a limit ordinal).

Remark. Let *Reg* be the class of regular cardinals and Λ the operation defined on $X \subseteq On$ by: $\Lambda(X) = \{\alpha \in X \mid |X \cap \alpha| = \alpha\}$. Then the α-weakly inacccessible cardinals are just the members of $\Lambda^\alpha(Reg)$, where:

$$\Lambda^0(Reg) = Reg$$
$$\Lambda^{\alpha+1}(Reg) = \Lambda(\Lambda^\alpha(Reg))$$
$$\Lambda^\mu = \bigcap_{\beta<\mu} \Lambda^\beta(Reg)$$

Proposition 2.12. If κ is the αth strongly inaccessible, where $\alpha < \kappa$, then the set of all regular cardinals below κ is nonstationary.

Proof. Let $(\kappa_i \mid i < \alpha)$ enumerate the inaccessibles below κ. We see that this is sequence is bounded by $\sup(\kappa_i \mid i < \alpha) < \kappa$. Take the set of all singular strong limit cardinals above this sup and less than κ, and proceed as the case of the least inaccessible. □

3 Mahlo

Definition 3.1.
κ is weakly Mahlo iff $\{\rho < \kappa \mid \rho \text{ is regular }\}$ is stationary in κ.

κ is strongly Mahlo iff $\{\alpha < \kappa \mid \alpha \text{ is strongly inaccessible}\}$ is stationary in κ.

Remark. Given that club sets are exactly the range of normal functions, an equivalent definition of weakly/strongly is that every normal function on κ has regular/strongly inaccesible fixed points.

Proposition 3.2.
(i) Weakly Mahlo cardinals are weakly inaccessible.
(ii) Strongly Mahlo cardinals are strongly inaccessible.

Proof.
Uncountability is straightforward in both cases.

Suppose a weakly Mahlo κ is not regular, and let $\mathrm{cf}(\kappa) = \gamma < \kappa$. Let $(\mu_i \mid i < \gamma)$ be cofinal in κ. WLOG, we may assume that $\mu_0 = \gamma + 1$. We look at the set C of limit points of this sequence other than κ: C must be club in κ. Hence C must contain a regular cardinal μ. But $\mu = \mathrm{cf}(\mu) < \mathrm{cf}(\kappa) < \mu < \kappa$. This argument also shows that strongly Mahlo cardinals are regular.

Let κ be weakly Mahlo. We now show that κ is a limit cardinal. Suppose not, then $\kappa = \delta^+$ for some δ. Then the set $C = \{x \in \kappa \mid \exists \alpha < \kappa \ x = \delta + \alpha\}$ is club in κ, but it contains no regular cardinal. Contradiction.
Now suppose κ is stronly Mahlo, and suppose for contradiction that κ is not strong limit. Then $2^\delta \geq \kappa$ for some $\delta < \kappa$. We consider again the set $C = \{x \in \kappa \mid \exists \alpha < \kappa \ x = \delta + \alpha\}$. This set is club in κ. This set cannot have strong limit cardinals, because all of its members are between δ and 2^δ. □

Remark. It follows from this and Prop 2.12 that if κ is weakly Mahlo, then κ is the κth weakly inacceesible. The converse might not hold: the least κ such that κ is the κth inaccessible is not Mahlo.

Proposition 3.3. κ is κ-weakly inaccessible.

Proof. Clearly, κ is a limit ordinal. So it suffices to show that for every $\alpha < \kappa$, κ is α-weakly inaccessible. We show this by induction on α.

Let $R = \{\rho < \kappa \mid \rho \text{ is regular}\}$. We define the following sequence of club sets of κ:

$$\begin{aligned} C_0 &= \kappa \\ C_{\alpha+1} &= \{x \in \kappa \mid \kappa \neq x \text{ is a limit point of } C_\alpha \cap R\} \\ C_\lambda &= \bigcap_{\alpha<\lambda} C_\alpha, \text{ for limit } \lambda \end{aligned}$$

$C_{\alpha+1}$ is club because $C_\alpha \cap R$ is stationary (hence unbounded in κ), and the limit points of an unbounded set is club. Hence by κ's Mahloness, $C_{\alpha+1}$ contains a regular cardinal, i.e., a $\alpha + 1$-weakly inaccessible. This shows that for all successor $\alpha < \kappa$, κ is α-inaccessible. The limit case is trivial. □

We may generalize Mahloness in a similar way.

Definition 3.4. κ is 0-weakly Mahlo iff κ is regular
κ is $\alpha + 1$-weakly Mahlo iff $\{\xi < \kappa \mid \xi \text{ is } \alpha\text{-weakly Mahlo}\}$ is stationary in κ
κ is δ-weakly Mahlo iff κ is α-weakly Mahlo for all $\alpha < \delta$, for limit δ.

Analogous to the Λ operation above, we can also define the Mahlo operation $M(X)$ to be: $M(X) = \{\alpha \in X \mid X \cap \alpha \text{ is stationary in } \alpha\}$. Then α-weakly Mahlo cardinals are just the members of $M^\alpha(Reg)$.

A Appendix

Claim A.1. (AC) successor cardinals are regular.

Proof. Let κ^+ be a successor cardinal and suppose for contradiction that $\mathrm{cf}(\kappa) = \lambda < \kappa$. Fix a cofinal sequence $(\kappa_\xi \mid \xi < \lambda)$, so $\kappa^+ = \bigcup_{\xi<\lambda} \kappa_\xi$. We show that $|\bigcup_{\xi<\lambda} \kappa_\xi| \leq \kappa < \kappa^+$ and thus arrive at a contradiction.

Since κ^+ is a cardinal, we know that $\kappa_\xi \leq \kappa$ for all $\xi < \lambda$. So for each $\xi \in \lambda$, we can fix a bijection f_ξ between $\kappa_x i$ and some $\mu \in \kappa$ (since there are many such bijections, here we use choice to pick one). We define a surjection f from $\kappa \times \kappa$ to $\bigcup_{\xi<\lambda} \kappa_\xi$ by setting $f(\alpha, \beta) = f_\alpha(\beta)$. Clearly, this function is surjective. Since $\kappa \times \kappa$ is well-orderable (even without choice), there is an injection f' from $\bigcup_{\xi<\lambda} \kappa_\xi$ to $\kappa \times \kappa$ (for each $x \in \bigcup_{\xi<\lambda} \kappa_\xi$, let $f'(x)$ be the least $(a, b) \in \kappa \times \kappa$ such that $f(a, b) = x$). This shows that $\kappa = \bigcup_{\xi<\lambda} \kappa_\xi \leq |\kappa \times \kappa| = \kappa < \kappa^+$. Which is a contradiction. □

Claim A.2. Let κ be an infinite cardinal, and let $(\gamma_\xi \mid \xi < \lambda)$ be cofinal in κ.
Then $\kappa = \bigcup_{\xi<\lambda} \gamma_\xi = \sum_{\xi<\lambda} |\gamma_\xi|$.

To prove this claim, we need the following:
Lemma. If λ is an infinite cardinal and $\kappa_i > 0$ for all $i < \lambda$, then

$$\sum_{i<\lambda} \kappa_i = \lambda \times \mathrm{sup}_{i<\lambda}\kappa_i$$

Proof. let $\kappa = \mathrm{sup}_{i<\lambda}\kappa_i$ and $\sigma = \sum_{i<\lambda} \kappa_i$. We want to show that $\sigma \leq \lambda \times \kappa$ and $\lambda \times \kappa \leq \sigma$.

$(\sigma \leq \lambda \times \kappa)$ $\kappa_i \leq \kappa$, and so $\sigma \leq \sum_{i<\lambda} \kappa \leq \lambda \times \kappa$.

$(\lambda \times \kappa \leq \sigma)$ note that $\lambda = \sum_{i<\lambda} 1 \leq \sigma$. Also note that $\kappa_i \leq \sigma$ for all $i < \lambda$. Putting these two together, we have $\sigma \geq \mathrm{sup}_{i<\lambda}\kappa_i = \kappa$. And so $\sigma \geq \lambda \times \kappa = \max\{\lambda, \kappa\}$. □

So to show our claim, it suffices to show that $\lambda \times \mathrm{sup}_{\xi<\lambda}|\gamma_\xi| = \kappa$. We now prove this.

Proof. We discuss two cases, (i) κ is a successor cardinal; (ii)κ is a limit cardinal.

(i) Since κ is a successor cardinal, κ is regular. Then by Claim A.1, $\lambda = \mathrm{cf}(\kappa) = \kappa$. Hence $\lambda \times \mathrm{sup}_{\xi<\lambda}|\gamma_\xi| = \lambda = \kappa$.

(ii) if κ is a limit cardinal, then for all $\mu < \kappa$, we have $\mu^+ < \kappa$. Since $(\gamma_\xi \mid \xi < \lambda)$ is unbounded in κ, it follows that $(|\gamma_\xi|)_{\xi<\lambda}$ is unbounded in κ also (because any bound to this sequence would also be a bound to the former sequence). Hence $\mathrm{sup}_{\xi<\lambda}|\gamma_\xi| = \kappa$ and so $\lambda \times \mathrm{sup}_{\xi<\lambda}|\gamma_\xi| = \kappa$ □