# Thinking of $C_1$ as a sharp

Jason Chen

The purpose of this small note is to present a point of view, according to which the largest $\Pi^1_1$ thin set is a kind of sharp. Nothing in here is new or original.

## 1 Countable objects transcending $L$

Recall the recurring motif between large cardinals and constructibility: large cardinals defeat constructible sets as a correct approximation of the true mathematical universe. The canonical example of this is Silver's work on sharps, where large cardinals makes $L$'s approximation of $V$ laughable:

**Theorem 1.** Assume there is a measurable cardinal, then

1. For every uncountable cardinal $\kappa < \lambda$, $L_\kappa$ is an elementary substructure of $L_\lambda$. In particular, $L_\kappa \prec L$.
2. There is a unique closed unbounded proper class $I$ of ordinals, such that for every uncountable cardinal $\kappa$ the following hold:
   (a) $|I \cap \kappa| = \kappa$
   (b) $I \cap \kappa$ is a set of indiscernibles for $(L_\kappa, \in)$
   (c) every $a \in L_\kappa$ is definable in $(L_\kappa, \in)$ from a finite increasing sequence of elements in $I \cap \kappa$

A consequence of this is that $L$ thinks every $\alpha \in I$ is an inaccessible cardinal. So there are many countable ordinals in $V$ that are mistaken by $L$ to be large cardinals.

Another failure of constructibility is that there are only countably many reals in $L$. This can be proved as follows:

**Corollary 2.** $\mathcal{P}(\omega) \cap L$ is countable

*Proof.* $\mathcal{P}(\omega) \cap L$ is definable in $L$ as "the unique $x$ whose members are exactly all the subsets of the set of natural numbers". By elementarity, this set is already definable in $L_{(\omega_1)^V}$. This entails that $\mathcal{P}(\omega) \cap L \in L_\beta$ for some $\beta < (\omega_1)^V$. To conclude that it is countable, we note that $|L_\beta| = |\beta| < (\omega_1)^V$. □

The feature of the kind of transcendence in Theorem 1 is that it's witnessed by a global object, namely elementary chain $L_\kappa \prec L_\lambda \prec ...$ and the club proper class $I \subseteq \mathrm{On}$.

It turns out that this transcendence is actually witnessed by a countable, simply definable object.

**Definition 3.** Let $0^\#$ denote that set of Gödel numbers of formulas $\varphi(v_1, ..., v_m)$ such that $L_{\aleph_\omega} \vDash \varphi(\aleph_1, ..., \aleph_m)$

By the works of Silver and Solovay, $0^\#$ is shown to have a simple definition in second order arithmetic: it is a $\Pi^1_2$ singleton. That is, there is a $\Pi^1_2$ formula $Z(x)$ such that $0^\#$ is the unique object satisfying it.

Moreover, $0^\#$ has a canonical/universal property: every well-founded remarkable Ehrenfeucht-Mostowski set of formulas will be identical to $0^\#$ (for definitions, see chapter 17 of Jech's Set Theory or §9 in Kanamori's The Higher Infinite).

Let us show Corollary 2. using this countable object.

*Another proof of countability of consturctible reals.* We show that every real in $L$ is computable from $0^\#$. Since there are only countably many programs, there can only be countably many such reals.

Now let $x \in \mathcal{P}(\omega) \cap L$. Observe that since $\mathsf{GCH}$ holds in $L$, $x \in L_{(\omega_1)^L} \subseteq L_{(\omega_1)^V}$. By 2(c) of Theorem 1., there is some formula $\varphi_x$ and some increasing sequence $i_1, ..., i_m$ of elements in $I \cap L_{(\omega_1)^V}$, such that

$$n \in x \Leftrightarrow (L_{(\omega_1)^V}, \in) \vDash \varphi_x(\bar{n}, i_1, ..., i_m)$$

(where we write $\bar{n}$ for the numeral for $n$; alternatively one could also substitute $n$ with the usual formula defining it.)

But since $L_{(\omega_1)^V}$ is an elementary substructre of $L_{\aleph_\omega}$, and since the elements of $I$ are all indiscernibles, we have

$$n \in x \Leftrightarrow (L_{\aleph_\omega}, \in) \vDash \varphi_x(\bar{n}, \aleph_1, ..., \aleph_m)$$

So to compute whether $n \in x$, we only need to search for the Gödel number of the formula $\varphi_x(\bar{n}, v_1, ..., v_m)$ in $0^\#$. This obviously can be done using a computer program with $0^\#$ as an oracle.

□

Finally, we note that it is fashionable nowadays to consider a countable object $M_0^\#$ ("the minimal active baby mouse" according to Schimmerling's The ABC's of Mice) that is Turing equivalent to $0^\#$.

$M_0^\#$ is a countable model of some weak fragment of set theory. The real coding this model is also a $\Pi^1_2$ singleton. It is similarly canonical/universal (this is the "minimal" part), in that it is the transitive collapse of the $\Sigma_1$-hull of the empty set on any active baby mouse. That is, if $M$ is any active baby mouse, then $\mathsf{Hull}_1^M(\emptyset)$ is isomorphic to $M_0^\#$.

The two preceding paragraphs are really just a rough summary of Schimmerling's The ABC's of Mice.

Taking stock: if there is a measurable cardinal, then there is a simply definable, canonical object that witnesses the failure of $L$ to approximate the true universe. Furthermore, it witnesses the failure of $L$ in a concrete manner (i.e., computing every real in $L$). The next section shows that something similar happens below the consistency strength of "$0^\#$ exists".

## 2 $\mathsf{PSP}(\mathbf{\Pi^1_1})$ and the set $C_1$

We say that a set $X \subseteq \mathbb{R}$ has the perfect set property, if it is either countable or has a perfect subset. Perfect set property is one of the regularity properties of sets of reals (two other famous ones are Lebesgue measurability and the property of Baire). For example, every Borel set has the perfect set property. We write $\mathsf{PSP}(\mathbf{\Pi^1_1})$ for the statement: every $\mathbf{\Pi^1_1}$ set has the perfect set property.

By results in descriptive set theory, $\mathsf{PSP}(\mathbf{\Pi^1_1})$ is revealed to be a large cardinal axiom.

**Theorem 4.** The following are equivalent

1. For all $x \in \mathbb{R}$, $(\omega_1)^V$ is an inaccessible cardinal in $L[x]$.
2. $\mathsf{PSP}(\mathbf{\Pi^1_1})$.
3. For all $x \in \mathbb{R}$, $\mathcal{P}(\omega) \cap L[x]$ is countable.

Recall that the existence of an inaccessible cardinal is consistent with $V = L$, where as "$0^\#$ exists" is not. Also, $0^\#$ has much higher consistency strength than inaccessibles, in that $\mathsf{ZFC}$ + "$0^\#$ exists" proves the consistency of $\mathsf{ZFC}$ + "there are proper class many inaccessibles".

**Question.** Can the failure of $L$ in the presence of $\mathsf{PSP}(\mathbf{\Pi^1_1})$ can be witnessed by some countable, simply definable, canonical object in a concrete manner?

It turns out the situation is surprisingly similar to that of $0^\#$. To show this, some recursion-theoretic machinery is needed. The following exposition follows the exposition in Chong & Yu's Recursion Theory. Another exposition can be found in Moschovakis's Descriptive Set Theory, chapters 4 and 5. The two expositions are essentially the same but in different languages.

**Definition 5.** For a real $x$, write $\omega_1^x$ for the supremum of the ordertypes of well-orderings that are computable in the oracle $x$. When $x$ is recursive, $\omega_1^x$ is known as $\omega_1^{CK}$, the Church-Kleene ordinal.

**Definition 6.** A set of reals is *thin* if it doesn't have a perfect subset.

**Definition 7.** $C_1 = \{x \in \mathbb{R} \mid x \in L_{\omega_1^x}\}$

The following sequence of lemmas are needed in order to characterize $C_1$.

**Lemma 8.** The relation $\{(x, y) \mid x \in \Delta^1_1(y)\}$ is $\Pi^1_1$.

**Lemma 9** (Spector-Gandy Theorem)**.** A set of reals $X$ is $\Pi^1_1(y)$ if and only if there is a $\Sigma_1$ formula $\varphi(u, v)$ such that for any reals $x$

$$x \in X \Leftrightarrow (L_{\omega_1^{x\oplus y}}[x \oplus y], \in) \models \varphi(x, y)$$

**Lemma 10** (Gandy Basis Theorem)**.** Every nonempty $\Sigma^1_1(x)$ set of reals has a hyperlow member, i.e., an element $y$ such that $y \leq_T \mathcal{O}^x$ and $\omega_1^{x\oplus y} = \omega_1^y$.

**Lemma 11** (Guaspari's Constructible Basis Theorem)**.** Every nonempty $\Pi^1_1$ set has a member $x$ such that $x \in L_{\omega_1^x}$.

**Theorem 12.** $C_1$ is a $\Pi^1_1$ thin set. That is, $C_1$ is $\Pi^1_1$ and does not have a perfect subset.

*Proof.* First, we use Spector-Gandy to calculate the complexity of $C_1$: if $x$ is a real, then

$$x \in C_1 \Leftrightarrow L_{\omega_1^x} \models (\exists\beta)(x \in L_\beta)$$

This give $C_1$ a $\Pi^1_1$ definition.

Second, we use the various basis theorems above to show that $C_1$ is thin. Suppose not, since perfect sets are branches on perfect trees, let $B$ be the set of (codes of) perfect trees whose branches are all in $C_1$:

$$B := \{T \mid T \text{ is a perfect tree whose branches are contained in } C_1\}$$

Note that $B$ is $\Pi^1_1$: coding a perfect tree is arithmetical, and to say the infinite branches of a tree are contained in $C_1$ is to say every real tracing an infinite branch is in $C_1$.

By Guaspari's theorem, there is some (code of a) tree $T \in B$ with $T \in L_{\omega_1^T}$. Notice also that $[T]$ is $\Sigma^1_1(T)$ ("there exists a real tracing a path in $T$..."). So, by Gandy Basis Theorem applied to the set $\{x \mid x \in [T] \wedge x \notin \Delta^1_1(T)\}$, there is some $x \in [T]$ that is hyperlow: i.e., $\omega_1^{x \oplus T} = \omega_1^T$.

To conclude: observe that $\omega_1^x \leq \omega_1^{x\oplus T} = \omega_1^T$. Now $x \notin \Delta^1_1(T)$, which is to say $x \notin L_{\omega_1^T}[T]$. Hence for this $x$, we have $x \notin L_{\omega_1^x}$, a contradiction. □

To show that $C_1$ is the largest $\Pi^1_1$ thin set, that is, if $X$ is a $\Pi^1_1$ thin set, then $X \subseteq C_1$, we use the lightface analogue of the Mansfield-Solovay theorem.

**Theorem 13** (Mansfield, Solovay)**.** Let $X$ be $\Pi^1_1$. If there is an $x \in X$ with $x \notin L_{\omega_1^x}$, then there is a perfect tree $T \in L$, whose branches all belong to $X$.

Observe that the contrapositive of Mansfield-Solovay theorem says that, for any $\Pi^1_1$ set $X$, if $X$ doesn't have a perfect subset, then every element $x \in X$ satisfies $x \in L_{\omega_1^x}$. That is, $x \in C_1$.

So, $C_1$ is simply definable (it's $\Pi^1_1$), and it has some universal property (it's the largest $\Pi^1_1$ thin set).

Finally, let us see that $C_1$ makes the set constructible reals countable in the presence of $\mathsf{PSP}(\boldsymbol{\Pi}^1_1)$. Some fine structure theory (projecta and master code) is needed for this.

**Theorem 14.** Every constructible real is computable from some element in $C_1$.

*Proof.* Let $x \in \mathcal{P}(\omega) \cap L$. Then $x \in J_{\alpha+1} \setminus J_\alpha$ for some $\alpha < (\omega_1)^L$. It follows that $x$ is $\Sigma_n(J_\alpha)$ for some $n$.

By fine structure theory, there is some real $z \in J_{\alpha+1} \setminus J_\alpha$ that is a $\Sigma_n$-master code for $J_\alpha$. In this case, to say $z$ is a $\Sigma_n$-master code for $J_\alpha$ is to say that (in the sense relevant to us) 1) $x$ is computable from $z$ and 2) there is some well-ordering of $\omega$ in ordertype $\alpha$ that is computable from $z$. Or in other words, $\omega_1^z > \alpha$ and $x \leq_T z$.

To conclude, notice that the master code $z \in J_{\alpha+1} \subseteq L_{\omega_1^z}$. And so $z \in L_{\omega_1^z}$, which is to say $z \in C_1$. □

**Corollary 15.** Assume $\mathsf{PSP}(\boldsymbol{\Pi}^1_1)$. Then $\mathcal{P}(\omega) \cap L$ is countable.

*Proof.* Under the assumption, every $\boldsymbol{\Pi}^1_1$ set is either countable or has a perfect subset. Since $C_1$ is $\Pi^1_1$ and doesn't have a perfect subset, it is countable. By the last theorem, every constructible real is computable from $C_1$. So there can be countably many constructible reals. □

To summarize, a countable, simply definable, canonical object $C_1$ is given, which in a concrete manner witnesses $L$'s failure to approximate $V$ in the presence of a large cardinal axiom with consistency strength much lower than "$0^\#$ exists".

**Question.** Is there a model of some weak theory that corresponds to $C_1$ like $M_0^\#$ corresponds to $0^\#$?