Mouse capturing at lower complexity

Jason Chen

Arguably, what makes an inner model "canonical" is the ability to capture truths about real numbers of certain complexity. Fancier inner models capture more complex reals. The purpose of this note is to record proofs of this so-called mouse capturing phenomena at the level of Δ^1_1 and Δ^1_2. This phenomenon has a few different names: e.g., in [Sar13] this was called Mouse Capturing, and in [Zhu16] this was called Mouse Set.

The phenomena take the following form: a) if a real number has a certain complexity, then it is "captured" by certain models of weak fragments of set theory b) no other real number gets in the inner model in question. Sargsyan describes this as "the spirit of canonicity in this context is that no random or arbitrary information is coded into the model. Every set in L has a reason for being in it".

The relevant measure of complexity here is the notion of "projective in a countable ordinal".

Definition 1. To say x is Δ^1_n in a countable ordinal α is to say that there is some Δ^1_n-formula/relation $D(u, v)$ such that (for a coding fixed in advance) whenever z is a code for α, we have $n \in x \Leftrightarrow D(n, z)$.

1 The Δ^1_2 case

Consider the "Gödel mouse", the L_α's. By Shoenfield absoluteness, coupled with an observation by Solovay, this mouse captures Δ^1_2 truths:

Theorem 2. A real $x \in \mathcal{P}(\omega)$ is Δ^1_2 in a countable ordinal if and only if it is in L.

This theorem is a little bit surprising, because it is possible for there to be many many countable ordinals that get mistaken by L as large cardinals. One would naively think that a real Δ^1_2 in one of those might code information that L cannot possibly know. But the theorem tells us that L is in some sense omniscient about Δ^1_2 facts.

Proof of Theoerm 2. If $x \in L$, then by the definition of L and GCH in L, x is definable over some countable level L_α, say with the formula $\varphi(v)$. So $n \in x$ if and only if there is some real (or equivalently for every real) coding a well-founded extensional structure of $\mathsf{KP} + V = L$, of height α, which additionally satisfies $\varphi(n)$. This is a Δ^1_2 statement with any code of α as parameter.

Conversely, assume x is Δ^1_2 in any code of α. If $L \vDash$ "α is countable." then α has a code in L. Using this code as parameter, one can then define x in L. By Shoenfield absoluteness, this indeed defines x (Δ^1_2 definitions of reals are absolute between V and L).

So without loss of generality, assume α is not countable in L. Let $\mathbb{P} \in L$ be the forcing in L to collapse α. And now force with $\mathbb{P} \times \mathbb{P}$. If $G \times H$ is generic for this forcing over L, then α has some code in $L[G]$ and $L[H]$, where one can carry out the Δ^1_2 definition using these codes. (Again, the definitions will be correct, by Shoenfield absoluteness).

The preceding paragraph shows that $x \in L[G] \cap L[H]$. An observation by Solovay (Lemma 2.5 in [Sol70]) says that $L[G] \cap L[H] = L$, thus concluding the proof. □

2 The Δ^1_1 case

The canonical inner model tied to hyperarithmetic theory is $L_{\omega_1^{CK}}$. Thinking of $L_{\omega_1^{CK}}$ as "hyperarithmetic mouse", we have the following:

Theorem 3. A real $x \in \mathcal{P}(\omega)$ is Δ^1_1 in a countable ordinal if and only if it is in $L_{\omega_1^{CK}}$.

Again, there are many many countable ordinals above ω_1^{CK}. One would naively think that such ordinals may provide additional information that takes one beyond hyperarithmetic reals. It turns out that hyperarithmetical truths are captured by the hyperarithmetic mouse.

The hyperarithmetical mouse case is slightly trickier. Of course, by Theorem 2 the real in question is in L. The difficulty is to give it a Δ^1_1 definition (equivalently, placing it in $L_{\omega_1^{CK}}$). To reduce the need for parameters, we use complexity calculations related to the category quantifier. One can equivalently use the analogous techniques for the measure quantifier. The proof presented here follows the analogous proof about Σ^0_α sets by Sami [Sam99]. First, a few lemmas are needed.

Lemma 4. The relation $x \in \Delta^1_1$ ("x is hyperarithmetic") is Π^1_1.

Lemma 5 ([Har68])**.** We say a real r codes a pseudo-well-ordering ($r \in \mathbf{pWO}$) iff every $\Delta^1_1(r)$ subset of Field(r) has a least element (in the relation coded by r). The relation $x \in \mathbf{pWO}$ is Σ^1_1.

Lemma 6. For any $r \in \mathbf{pWO}$ such that $\omega_1^r = \omega_1^{CK}$, the relation coded by r has a recursive isomorphic copy.

Lemma 7 (Gandy Basis Theorem)**.** Every nonempty Σ^1_1 set of reals has a hyperlow element, i.e., a real r such that $\omega_1^r = \omega_1^{CK}$.

Lemma 8 (Category Quantifier Calculation, see [Kec12], Theorem (29.22))**.** If $\varphi(x, y)$ is Δ^1_1, then so is "there are co-meagerly many $y \in S_\infty$ such that $\varphi(x, y)$"

Here's the key idea of the proof: for $x \in \Delta^1_1(\alpha)$, we want to provide a Δ^1_1 definition. We observe that coding α is a common property relative to S_∞ (the set of bijections $f : \omega \to \omega$) in the following sense: if r codes a well-ordering relation on ω of ordertype α, then any bijection on ω will give rise to another such well-ordering (which we shall write as $f * r$). This will allow us to reduce Δ^1_1-in-a-code to Δ^1_1-comeagerly-many-in-S_∞, and with the help of the category quantifier calculation lemma, we can dispense the need for parameters. The trick here is to do this to possibly some other member of set of counterexamples to the theorem, not necessarily the particular x that we started with.

Proof of Theorem 3. Let x be Δ^1_1 in a countable ordinal, say it's defined by the Δ^1_1 relation $\varphi(n,r)$. Suppose towards a contradiction that $x \notin \Delta^1_1$.

Begin by noticing that we have: fix r a code for α, then for all and $n \in \omega$,

$$n \in x \Leftrightarrow (\forall f \in S_\infty)(\varphi(n, f * r)) \Leftrightarrow (\exists f \in S_\infty)(\varphi(n, f * r))$$

Now consider the set of x-like counterexamples

$$A = \{(y,v) \mid y \notin \Delta^1_1 \wedge v \in \mathbf{pWO} \wedge \text{ for co-meagerly many } f \in S_\infty, (\forall n \in \omega)(n \in y \Leftrightarrow \varphi(n, f*v_0))\}$$

By the lemmas above, A is Σ^1_1, and since we assumed (x,r) is such a counterexample, A is not empty. Now apply Gandy Basis to find some (y_0, v_0) such that $\omega_1^{(y_0,v_0)} = \omega_1^{CK}$. In particular, $\omega_1^{v_0} = \omega_1^{CK}$. By Lemma 6, there is a recursive real w coding an isomorphic relation as the one coded by v_0.

But now it is easy to define y_0: for any natural number n:

$$n \in y_0 \Leftrightarrow \text{ for co-meagerly many } f \in S_\infty, \varphi(n, f * w)$$

To conclude, observe that since w is recursive, the above definition is Δ^1_1, contradicting the choice of y_0. □

References

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