No Borel 2-coloring of the irrational rotation graph: a simple application of Borel codes and absoluteness

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1 ${}^{\mathbb{Z}}\omega$ and irrational rotations

Consider the set of $\mathbb{Z}$-chains of natural numbers modded out with constant shifts. In other words, place the equivalent relation $\sim$ on ${}^{\mathbb{Z}}\omega$ by $f \sim g$ iff there is some $k \in \mathbb{Z}$ such that for all $z \in \mathbb{Z}$, $f(z) = g(z+k)$ (as a convention, we call g a k shift of f when this holds).

This structure is interesting since it is a proxy to study the dynamical system of irrational rotations $T_\theta : S^1 \to S^1, x \mapsto xe^{2\pi i\theta}$; or equivalently $T_\theta : [0,1) \to [0,1), x \mapsto x + \theta \mod 1$. For each single point x, its movement is a $\mathbb{Z}$-chain $(..., T^{-2}(x), T^{-1}(x), x, T(x), T^2(x), ...)$. If we just care about the orbits themselves and not where they originate, then this is where the "mod constant shift" comes in: by identifying chains with the same trajectories, we are practically forgetting the origins.

It turns out that in many dynamical/measure/category aspects ${}^{\mathbb{Z}}\omega/\sim$ is very much isomorphic to irrational rotations. One may think of the naturals in the case of ${}^{\mathbb{Z}}\omega$ as marking what rational interval a point has fallen into, but that's not very helpful for what follows. Rather, having said something that hopefully explains why it is interesting, we will stick with ${}^{\mathbb{Z}}\omega/\sim$ as they are and only fall back on the rotation analogy when it is useful.

(Having said the above, it nevertheless happens to be useful to think about rotations now) If T is an irrational rotation, we may think of the points as the vertices in a graph and x, y are related by an egde if either $T(x) = y$ or $T(y) = x$. A simple irrationality consideration (think in terms of $x \mapsto x + \theta \mod 1$) shows that this graph is acyclic, i.e., there are no repeats in $(..., T^{-2}(x), T^{-1}(x), x, T(x), T^2(x), ...)$, and each vertx has exactly two neighbors. So in this graph, the connected components are excatly the orbits. Such graphs and their combinatorial properties are the object of study of the recent field of descriptive combinatorics. For example, we may ask: is there a 2-coloring of this graph?

2 The graph that we are interested in

Let us try to copy this question onto ${}^{\mathbb{Z}}\omega$. By analogy, each $f : \mathbb{Z} \to \mathbb{N}$ is supposed to represent the orbit of some point, and we wish to use the points on this chain as vertices and put edges between them and ultimately think about their coloring properties. A small issue:

this makes things a little messy, in that we need to distinguish the points on different chains and hence our vertices will look something like (n, z, f) to represent the fact that $f(z) = n$. This will require our space to be $\omega \times \mathbb{Z} \times {}^{\mathbb{Z}}\omega$, which is not a serious problem but could make for bloated notations.

Luckily, this won't be necessary for our present question. Recall that a 2-coloring is a way of coloring the vertices in two colors such that adjacent vertices receive distinct colors. In a $\mathbb{Z}$-chain, each such coloring can be determined by what color the origin (i.e., $f(0)$) gets, since the 2 colors just alternate. So we may pretend the vertices are $f : \mathbb{Z} \to \mathbb{N}$.

What about the edges? In the rotation example, each point is adjacent to its T-successor and predecessor. But here we've identified each origin-aware $\mathbb{Z}$-chain with its origin, so successor and predecessor internal to this chain don't make sense any more. But recall that each origin-agnostic orbit is ultimately represented by an equivalence class of such chains, and again the alternating nature of 2-colorings here tells us that f and g should receive the same color if $f \sim g$ and g is an even-integer shift from f.

Similarly, f and g should receive different colors if $f \sim g$ (i.e., they represent the same orbit) and g is an odd-integer shift from f. For example, if $f \sim g$ and the origin of g is, say, $f(1)$, then f should get red if and only if g gets blue. So the only obstruction to a 2-coloring is odd-integet shifts.

Taking stock, we are interested in the following graph:

Definition 1. The (proxy) irrational rotation graph $\mathcal{G}$ has as vertex set the functions $f : \mathbb{Z} \to \mathbb{N}$, and f and g are adjacent iff one is an odd-integer shift of the other.

Definition 2. A 2-coloring on $\mathcal{G}$ is a function $c : {}^{\mathbb{Z}}\omega \to \{\text{red, blue}\}$ such that if f is adjacent to g, then $c(f) \neq c(g)$. Such a coloring is Borel if it's Borel as a function, or equivalently, if the sets that gets each color are Borel sets respectively.

3 Results

The first question is whether a 2-coloring exists. Using the axiom of choice we can show that they do.

Observation. *Assuming the axiom of choice, a 2-coloring exists for $\mathcal{G}$.*

Proof. In effect, we will use the axiom of choice to impose an origin for each origin-agnostic orbit. That is, from each equivalence class $[f]_\sim$, choose a representative f and color it red. Then color the $+1, -1$ shifts blue, the $+2, -2$ shifts red, the $+3, -3$ shifts blue, etc. $\square$

Objects constructed using AC tend to be pathological. In particular they tend not be Borel/Lebesgue-measurable/Baire-measurable. This is true in our case too.

Theorem 3. *There is no Borel 2-coloring for $\mathcal{G}$.*

Proof. If c is a Borel 2-coloring, then the sets

$$A := \{f \mid c(f) = \text{red}\}$$
$$B := \{f \mid c(f) = \text{blue}\}$$

are both Borel.

Hence the following sentence has complexity Π^1_1 with respect to the Borel codes of A, B (that's how the quotation marks are supposed to be paraphrased away):

$$\forall f, g(\text{“}f \in A\text{”} \vee \text{“}f \in B\text{”}) \wedge \neg(\text{“}f \in A \cap B\text{”}) \wedge$$
$$g \text{ is an odd-integer shift of } f \Rightarrow (\text{“}f \in A\text{”} \iff \text{“}g \in B\text{”})$$

By Mostowski absoluteness, this sentence is absolute between a ground model and forcing extensions. We will use this absoluteness property and the symmetry of Cohen forcing to prove that no Borel coloring exists.

In this context, the forcing poset to add a Cohen real is the set $\mathbb{C}$ of finite partial functions $p : \mathbb{Z} \to \omega$. After forcing to add a Cohen real g, in $V[g]$ the same sentence holds true by Mostowski absoluteness. So the Cohen real g must fall into either $A^{V[g]}$ or $B^{V[g]}$, that is, the interpretations of the Borel codes for A, B in $V[g]$.

Without loss of generality, let's assume $g \in A^{V[g]}$. This is going to be forced by some finite condition p, letting a be the Borel code for A and Γ the canonical name for the generic filter:

$$p \Vdash \Gamma \text{ is in the set coded by } \check{a}.$$

The $+1$ shift map $i : p \to p'$ with $p(z) = p'(z+1)$ is an automorphism on $\mathbb{C}$. Any such automorphism induces a map i between names that fixes all the check names (canonical names for ground model elements), such that $p \Vdash \varphi(\vec{x}) \iff i(p) \Vdash \varphi(\overrightarrow{i(x)})$ and also g is generic iff $i\text{”}g$ is (all this can be found in Kunen, Chapter VII Section 7).

So shift p by $+1$, so we have

$$i(p) \Vdash \Gamma \text{ is in the set coded by } \check{a}.$$

But notice in particular that the $+1$ shift g' of g is a generic extending $i(p)$, and moreover $V[g'] = V[g]$. So $V[g] \vDash g' \in A^{V[g]}$, contradicting the earlier fact that odd-integer shifts must also get different colors in $V[g]$.

□

With a bit more care, one can show that there can be no Baire-measurable 2-coloring. But this involves a more careful analysis of the relationship between Cohen reals and Baire category notions. Similarly it is also true that there is no measurable 2-coloring. The quickest proof of this goes through the ergodicity of $T \circ T$ and derives a contradiction by saying that the two color sets must be both null or conull at the same time.