PROPERNESS OF DEFINABLE FORCING NEED NOT BE PROJECTIVE

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Abstract. We give a negative consistency result for the projective definability of properness when definability is required of the domain and ordering of a forcing notion. Starting in L, we construct a uniformly closed family of partial orders and a forcing extension with no new countable sequences of ordinals in which the set of parameters giving proper partial orders does not belong to $L(\mathbb{R})$. The construction uses coanalytic incompatibility graphs on codes for club-shooting conditions. Finite independent sets in these graphs give analytic presentations, which can be replaced by closed presentations without changing the forcing. An almost-disjoint coding refinement gives the same nonprojectivity conclusion for closed presentations whose generic filter is recovered from a single generating real by a fixed Borel relation. We do not require analytic incompatibility or separativity, and make no assertion under additional large-cardinal hypotheses. This work is done in with the help of GPT-6.

1. Main Results

Zapletal asks whether properness of definable, for example analytic, forcing notions is a projective property [5, Question 2.1.4]. This question is subject to the specific requirements imposed on the presentation of the forcing notion. Analyticity of the domain and ordering does not imply analyticity of incompatibility. The latter is included in the definition of a Suslin forcing used, for example, in [3, Definition 3.2]. We consider the formulation involving the domain and ordering alone and provide a negative consistency result to Zapletal's question.

A *closed presentation* is a partial order $(P, \leq)$ such that P is a closed subset of a fixed zero-dimensional Polish space and $\leq$ is closed in the square of that space. All the spaces below can be identified with closed subspaces of ω^ω. The word "closed" here is topological; it does not mean countably closed as a forcing. We do not assume that a presentation is separative. Zapletal likewise permits nonseparative presentations in the discussion preceding his question [5, Chapter 2, p. 5].

A family $(\mathbb{P}_a : a \in \omega^\omega)$ is *uniformly closed* if there are closed sets $D \subseteq \omega^\omega \times \omega^\omega$ and $O \subseteq \omega^\omega \times \omega^\omega \times \omega^\omega$ such that $\mathbb{P}_a = (D_a, O_a)$, with O_a a partial ordering of the nonempty set D_a. We regard a as a parameter for a fixed definition. Equivalently, we may pass continuously from a to a pair of ordinary closed-set codes for D_a and O_a.

We use the standard definition of properness. If θ is sufficiently large, a condition $q \leq p$ is an *M-master condition*, for a countable elementary submodel $M \prec H_\theta$ with $P, p \in M$, if

$$q \Vdash \dot{G} \cap M \text{ is } (P \cap M)\text{-generic over } M.$$

Equivalently, for every dense set $D \subseteq P$ with $D \in M$, the set $D \cap M$ is predense below q. The forcing P is *proper* if, for every sufficiently large regular θ, every countable $M \prec H_\theta$ with $P \in M$, and every $p \in P \cap M$, there is an M-master condition $q \leq p$.

Theorem 1.1. *Assume $V = L$. There are a uniformly closed family $(\mathbb{P}_a : a \in \omega^\omega)$ and a forcing $\mathbb{S}$ adding no countable sequences of ordinals such that, in every $\mathbb{S}$-generic extension W,*

$$B^W = \{a \in \omega^\omega : W \models \mathbb{P}_a \text{ is proper}\} \notin L(\mathbb{R})^W.$$

The closed sets defining the family belong to L and are interpreted from the same codes in W. In particular, $W \models \mathrm{CH}$, and B^W is not projective.

Corollary 1.2. *Relative to* ZFC*, it is consistent with* ZFC + CH *that the set of real codes for proper partial orders with closed domain and closed ordering relation is not projective. In that model the code set does not belong to* $L(\mathbb{R})$*.*

The family in Theorem 1.1 is fixed before forcing. Its interpretation, as a family of sets and relations, does not change in the extension. A new subset of ω_1 is recovered from the properness of a constructible sequence of its members. The use of an entire sequence, rather than a single instance of nonabsoluteness, is what excludes projective definitions with arbitrary real parameters.

The forcings used for the main theorem add no reals and hence are not real forcings in the sense of [5, Definition 2.1.5]. The final section, due entirely to GPT-6 Astra, shows that the construction can be refined so that the generic extension is generated by a single real, from which the generic filter is recovered by a Borel relation, while the same properness coding is preserved.

Theorem 1.3. *Relative to* ZFC*, it is consistent with* ZFC + CH *that there is a uniformly closed family* $(\mathbb{R}_a : a \in \omega^\omega)$ *with the following properties. Each* $\mathbb{R}_a$ *adds a real* z *generating the entire forcing extension, and a fixed Borel relation* $\chi(p, z)$*, in the chosen coding of conditions, recovers its generic filter. For every nonempty proper basic clopen set* $O \subsetneq 2^\omega$ *there are conditions forming a two-element maximal antichain whose recovery relations are exactly* $z \in O$ *and* $z \notin O$*. Nevertheless,*

$$\{a : \mathbb{R}_a \text{ is proper}\} \notin L(\mathbb{R}).$$

In particular this set is not projective.

The recovery and basic-open-set requirements are those in Zapletal's definition of a real forcing [5, Definition 2.1.5]. The proof of Theorem 1.1 and Corollary 1.2 is independent of the almost-disjoint coding construction used to prove Theorem 1.3 in Section 7.

We use standard facts about forcing and constructibility as in [1], and the closed-projection characterization of analytic sets as in [2, Chapter III]. The coding and preservation arguments needed for the main construction are given below. Section 6 discusses the scope of the conclusion, and Section 7 gives the single-real refinement.

2. Presentations and finite independent sets

We order forcing notions so that smaller conditions are stronger. Preorders will be used as intermediate presentations. Compatibility and forcing equivalence always refer to their separative quotients.

Lemma 2.1. *Suppose* $\pi : P \to Q$ *is a surjective, order-preserving map between preorders. Assume that* p *and* p' *are compatible exactly when* $\pi(p)$ *and* $\pi(p')$ *are compatible, and that whenever* $q \leq \pi(p)$ *there is* $p' \leq p$ *with* $\pi(p') \leq q$*. Then* π *induces an isomorphism of separative quotients. In particular,* P *and* Q *are forcing-equivalent.*

Proof. Recall that $p \leq^* p'$ means that every extension of p is compatible with p'. If $\pi(p) \leq^* \pi(p')$, order preservation and reflection of compatibility give $p \leq^* p'$. Conversely, if $\pi(p) \not\leq^* \pi(p')$, choose $q \leq \pi(p)$ incompatible with $\pi(p')$ and lift q to an extension of p. That extension is incompatible with p'. Thus π preserves and reflects $\leq^*$, and surjectivity gives the assertion. □

For a Polish space X, let $\mathrm{Fin}(X) = \coprod_{k<\omega} X^k$, with the disjoint-union topology. We use finite sequences rather than a space of unordered finite sets. Repetitions and the order of entries are immaterial to the forcing. For $u \in \mathrm{Fin}(X)$ write $\mathrm{supp}(u) = \mathrm{ran}(u)$, and put $u \preccurlyeq v$ if $\mathrm{supp}(u) \supseteq \mathrm{supp}(v)$. This is a closed preorder on $\mathrm{Fin}(X)$. On each pair of components it is a finite intersection of finite unions of equality conditions.

Let $E \subseteq X^2$ be a symmetric irreflexive coanalytic relation. Define

$$P(E) = \{u \in \mathrm{Fin}(X) : (\forall x, y \in \mathrm{supp}(u))\ \neg E(x, y)\},$$

ordered by $\preccurlyeq$. The domain is analytic, and two conditions are compatible exactly when the concatenation of their sequences belongs to $P(E)$. These assertions hold uniformly for coanalytic families of graphs.

Lemma 2.2. *Let Q be a partial order with a weakest condition 1_Q. Suppose every nonempty finite pairwise compatible subset of $Q \setminus \{1_Q\}$ has a strongest member. Let $V_0 \subseteq X$ and let $d : V_0 \to Q \setminus \{1_Q\}$ be onto. Suppose*

$$E(x, y) \iff x, y \in V_0 \text{ and } d(x) \perp_Q d(y)$$

is coanalytic. Then $P(E)$ is forcing-equivalent to Q.

Proof. For $u \in P(E)$ let $\pi(u)$ be the strongest member of $\{d(x) : x \in \operatorname{supp}(u) \cap V_0\}$ if this set is nonempty, and let $\pi(u) = 1_Q$ otherwise. This is well-defined and order-preserving. It is onto, since a one-element sequence represents every nontrivial condition, while the empty sequence represents 1_Q.

If u, v are compatible, their concatenation shows that $\pi(u), \pi(v)$ are compatible. Conversely, a common extension of $\pi(u), \pi(v)$ extends every decoded condition occurring in either sequence. Hence all pairs in their union are E-independent, including the pairs involving points outside V_0, and u, v are compatible. Finally, if $q \leq \pi(u)$ and $q \neq 1_Q$, append to u a point $x \in V_0$ with $d(x) = q$. This gives an extension mapped to q. The case $q = 1_Q$ needs no change to u. Apply Lemma 2.1. $\square$

The points outside V_0 are not discarded. They are isolated vertices of E and impose no restriction on the separative quotient. This is why the validity of a code need not be an analytic condition on the domain of the resulting forcing.

Lemma 2.3. *Let I, Y be zero-dimensional Polish spaces, let $A \subseteq I \times Y$ be analytic with nonempty sections, and let $\preccurlyeq$ be a closed preorder on Y. There is a uniformly closed family of partial orders $(\widehat{P}_a : a \in I)$ such that $\widehat{P}_a$ is forcing-equivalent to $(A_a, \preccurlyeq)$ for every a.*

Proof. Choose a closed set $F \subseteq I \times Y \times \omega^\omega$ whose projection onto $I \times Y$ is A. A condition of $\widehat{P}_a$ is (u, w, k) with $(a, u, w) \in F$ and $k < \omega$. Put

$$(u, w, k) \leq (v, z, l) \iff (u, w, k) = (v, z, l) \text{ or } (u \preccurlyeq v \text{ and } k > l).$$

The domain is closed. The ordering, including its domain requirements, is a finite union of closed sets. It is reflexive and transitive, and the strict inequality of the integer coordinates gives antisymmetry.

Projection to u is onto A_a and order-preserving. A common extension in A_a lifts to a common extension in $\widehat{P}_a$ by choosing a closed-set witness and a sufficiently large integer coordinate. The same observation gives the lifting property for an extension of one projected condition. Compatibility in the other direction follows by projection. Apply Lemma 2.1.

Products and countable disjoint unions used here admit fixed closed embeddings into ω^ω. After making these identifications, the domains and ordering relations are jointly closed in the parameter and condition coordinates. $\square$

In particular, adjoining closed-set witnesses does not add an independent generic object. The projection in Lemma 2.3 preserves the entire separative quotient. The integer coordinate is only needed to obtain a partial order rather than a preorder with multiple equivalent names.

3. Constructible stationary sets and countable models

For the next two sections work in L, and put $\kappa = \omega_1^L$. Use a canonical well-order $<_L$ which orders sets by their first constructibility stage before ordering the sets appearing at one stage. Its restrictions to the relevant initial segments are absolute. For every infinite $\beta < \kappa$, let $e_\beta : \omega \to \beta$ be the $<_L$-least bijection. For $\xi < \kappa$ and $n < \omega$ define

$$S_{\xi,n} = \{\beta < \kappa : \beta > \max\{\xi, \omega\} \text{ and } e_\beta(n) = \xi\}.$$

Lemma 3.1. *There are $n_* < \omega$ and distinct ordinals ξ_*, ξ_i for $i < \kappa$ such that $S_* = S_{\xi_*,n_*}$ and $S_i = S_{\xi_i,n_*}$ are stationary. All these stationary sets are pairwise disjoint.*

Proof. For each ξ, the union of $S_{\xi,n}$ over $n < \omega$ contains a tail of κ. By countable completeness of the nonstationary ideal, some $S_{\xi,n}$ is stationary. Assign such an n to each ξ. Since κ is regular and uncountable, one value n_* is assigned to κ many ordinals. For fixed n_* the sets S_{ξ,n_*} are pairwise disjoint, because $e_\beta(n_*)$ has only one value. Reserve one index and enumerate κ many of the remaining indices. $\square$

This is the usual stationary-set splitting argument applied to canonical constructible enumerations. The canonical choice is needed for the definability of the individual sets; the selection of the stationary indices is made in L and need not itself be projective.

Let T be the following fixed countable first-order theory: ZFC without Power Set, with full Separation and Collection (and hence Replacement), together with $V = L$ and the assertion that every set is countable. Choice is retained. Countable here allows finite sets.

Lemma 3.2. *The structure $H^L_\kappa = L_\kappa$ satisfies T. Every transitive model of T is of the form L_α. If such a model contains an infinite ordinal β, its internally computed $<_L$-least bijection from ω to β is the true constructible least bijection.*

Moreover, for any finitely many objects in H^L_κ, there is in L a countable well-founded extensional model of T whose collapse contains and fixes those objects as distinguished elements.

Proof. First, $H^L_\kappa = L_\kappa$. One inclusion follows because every L_γ, for $\gamma < \kappa$, is countable in L. For the other, given a hereditarily countable constructible set x, take in L a countable elementary submodel of a sufficiently large L_θ containing the transitive closure of $\{x\}$ pointwise. Condensation gives a collapse L_α with $\alpha < \kappa$, and the collapse fixes x.

H^L_κ satisfies the usual axioms of set theory without Power Set. For Collection, a set in H^L_κ is countable in L; choosing one hereditarily countable witness for each of its elements still gives a hereditarily countable set. Separation and Choice are verified in the same way. Every set of this structure has a counting function in the structure. Its equality with L_κ also gives $V = L$ internally.

Now let M be transitive and satisfy T, and let $\alpha = \mathrm{Ord} \cap M$. The construction of L_γ is absolute for $\gamma \in M$: at successor stages first-order satisfaction for the set-sized structure L_γ is absolute, and at limit stages one takes unions. Thus $L^M_\gamma = L_\gamma$ for every $\gamma \in M$. Internal $V = L$ then gives $M = L_\alpha$. The canonical constructible ordering on its elements is likewise the true restriction of $<_L$.

If $\beta \in M$ is infinite, the assertion that every set is countable supplies a bijection from ω to β in M. Its internally least such bijection is a genuine bijection. Every constructible object earlier in $<_L$ is already in L_α, since the ordering is stage-respecting. Thus no earlier bijection has been omitted, proving the assertion about e_β.

Finally, take a countable $N \prec H^L_\kappa$ containing the given objects. Each of their transitive closures is countable in H^L_κ. An enumeration belonging to N shows that these transitive closures are subsets of N. Hence the transitive collapse fixes each given object. The collapsed model satisfies T and has a countable real code in L. $\square$

No countable transitive model of full ZFC is needed here. In particular, the existence of these codes entails no inaccessible cardinal assumption. Power Set is absent from T.

4. Coanalytic incompatibility and the closed family

For $T \subseteq \kappa$, let $\mathrm{CU}(T)$ consist of the countable closed bounded subsets of T, including the empty set, ordered by end-extension. The empty set is the weakest condition. For nonempty conditions,

$$c \leq d \iff c \cap (\max(d) + 1) = d.$$

Two nonempty conditions are compatible exactly when one end-extends the other. Therefore a nonempty finite pairwise compatible family has a strongest member.

We now give a uniform coanalytic graph representing incompatibility for $\mathrm{CU}(\kappa \setminus S_{\xi,n})$. The parameter is a pair $a = (n, r)$, where r is a real coding a well-order of type ξ. We use a fixed coding of binary relations on subsets of ω; arbitrary reals are allowed as parameters, whether or not they code well-orders. The parameter space is identified with ω^ω.

Let $X = \omega^\omega$ be a space of raw codes for countable relational structures with finitely many distinguished elements. For a code x, write M_x for its coded structure. Say that $\mathrm{Valid}(a, x)$ holds if M_x is well-founded and extensional, satisfies T, and has correctly distinguished elements for the parameter r, an ordinal ξ, a nonempty set c, and its maximum b. Require in addition that M_x verifies that r is isomorphic to the ordinal ξ and that c is a closed bounded set of ordinals, with maximum b, avoiding the set given by the formula defining $S_{\xi,n}$.

The requirement that the distinguished real is the actual parameter r means that all of its digits are correct, not merely that the model regards it as a real. This is a Borel condition on a code. Likewise, satisfaction of the fixed first-order theory and the remaining assertions is Borel on countable structures. Consequently, Valid is coanalytic: the sole additional complexity is external well-foundedness of the membership relation.

The isomorphism to the distinguished ordinal is required internally, not just the assertion that every internally available subset of the coded order has a least element. In a valid, collapsed model that isomorphism is an actual isomorphism. In particular, an ill-founded parameter has no valid codes.

For a valid x, let c_x be the actual closed bounded set obtained by collapsing its distinguished element c. By Lemma 3.2, c_x really belongs to $\mathrm{CU}(\kappa \setminus S_{\xi,n}) \setminus \{\varnothing\}$. For every $\beta \in c_x$ above $\max\{\xi, \omega\}$, the model's canonical e_β is the true one. Conversely, every nonempty condition in this club-shooting forcing has a valid code: apply Lemma 3.2 to r and the condition. The collapse fixes both.

Lemma 4.1. *There is a coanalytic relation $E(a, x, y)$ whose sections E_a are symmetric irreflexive graphs and which, for valid parameters $a = (n, r)$ in L, satisfies*

$$E_a(x, y) \iff \mathrm{Valid}(a, x) \wedge \mathrm{Valid}(a, y) \wedge c_x \perp_{\mathrm{CU}(\kappa \setminus S_{\xi,n})} c_y.$$

Invalid codes are isolated vertices of E_a.

Proof. From a raw code x, extract the coded relation on the elements which are predecessors of its distinguished maximum b, together with b itself. Mark those elements which belong to its distinguished c. For a valid code this is a well-order of type $b + 1$, marked exactly by c_x.

Let $C(x, y)$ assert that there is an isomorphism from the marked order extracted from y onto a downward-closed initial segment of the marked order extracted from x. Preservation of the markings is required in both directions on that initial segment. A candidate isomorphism is coded by a real, and all its required properties are Borel. Thus C is analytic on all raw codes. For valid codes it asserts precisely that c_x end-extends c_y.

Define

$$E(a, x, y) \iff \mathrm{Valid}(a, x) \wedge \mathrm{Valid}(a, y) \wedge \neg C(x, y) \wedge \neg C(y, x).$$

This is coanalytic and symmetric. It is irreflexive because the identity isomorphism witnesses $C(x,x)$ for every valid code. The stated interpretation on valid codes follows from the compatibility criterion for end-extension. □

Proposition 4.2. *There is a uniformly closed family $(\mathbb{P}_a : a \in \omega^\omega)$ such that, for every well-order code $r \in L$ of type $\xi < \kappa$ and every $n < \omega$,*

$$\mathbb{P}_{(n,r)} \simeq \mathrm{CU}(\kappa \setminus S_{\xi,n})$$

in L. The same forcing equivalence holds in every extension of L adding no countable sequences of ordinals, where the club-shooting forcing is computed in that extension.

Proof. Use the coanalytic graph of Lemma 4.1. The set of pairs (a,u) with $u \in P(E_a)$ is analytic. Its ordering is the restriction of the fixed closed support-inclusion preorder on $\mathrm{Fin}(X)$. Apply Lemma 2.3 to obtain the uniformly closed family $\mathbb{P}_a$. All sections are nonempty, since the empty sequence belongs to $P(E_a)$, including for invalid parameters.

For a valid parameter in L, the decoding map from its valid codes is onto the nonempty club-shooting conditions. These conditions satisfy the hypothesis about finite compatible families in Lemma 2.2. That lemma gives the required forcing equivalence.

Now let W be an extension adding no countable sequences of ordinals. It has the same reals and the same ω_1 as L. The code sets, graphs, finite sequences, and closed presentations are unchanged. Well-founded collapses of the old codes are unchanged as well. Every countable closed bounded subset of κ in W is already in L, since a new one would have a new countable enumeration of ordinals. Thus the club-shooting partial order computed in W is exactly the old one. The projection and lifting arguments from Lemmas 2.2 and 2.3 still apply in W. □

5. Changing properness without adding countable sequences

We record the club-shooting facts used in the proof. They are the ω_1 case of the standard distributivity and preservation arguments for this forcing; see also [4, Section 2.1] for more general versions. Here the proofs need only stationarity, not a separate hypothesis about fat stationary sets.

Lemma 5.1. *Let $T \subseteq \omega_1$ be stationary. Then $\mathrm{CU}(T)$ adds a club subset of T, adds no countable sequences of ordinals, and preserves every stationary subset of T. Moreover,*

$$\mathrm{CU}(T) \text{ is proper} \iff T \text{ contains a club.}$$

Proof. First prove that no countable sequence of ordinals is added. Fix a condition p and a name $\dot{f}$ for such a sequence. Choose a large regular θ and a countable $N \prec H_\theta$ containing the relevant parameters, with $\delta = N \cap \omega_1 \in T$. Take an increasing sequence of ordinals from N cofinal in δ. Externally construct a descending sequence of conditions belonging to N, starting below p, deciding $\dot{f}(n)$ at step n, and having maxima cofinal in δ. The existence of the next condition follows by elementarity; the unboundedness of T permits increasing the maximum as required. The union of this sequence, with δ adjoined, is a condition extending every member and deciding the whole sequence. Such conditions are dense below every p.

It follows in particular that ω_1 is preserved. Conditions with maximum above any prescribed ordinal are dense, so the generic union is unbounded. If an ordinal is a limit point of the generic union, any condition in the filter with larger maximum witnesses that this limit point is in the union. Thus the union is closed and contained in T.

For stationary-set preservation, let $S \subseteq T$ be stationary, let $\dot{D}$ be a name for a club in ω_1, and start below an arbitrary condition. Choose N as above with $\delta = N \cap \omega_1 \in S$. In the descending construction, additionally force ordinals from N into $\dot{D}$ cofinally in δ. The final lower bound forces $\delta \in \dot{D} \cap S$. Hence S remains stationary.

If $C \subseteq T$ is club, the conditions whose maxima belong to C form a dense countably closed suborder. For a descending sequence in this suborder, either the conditions eventually stabilize or the supremum of their maxima belongs to C; adjoining that supremum to their union gives a lower bound in the suborder. Thus $\mathrm{CU}(T)$ is proper. If T contains no club, its complement is stationary, and the generic club in T destroys that stationarity. Since proper forcing preserves stationary subsets of ω_1, $\mathrm{CU}(T)$ cannot be proper. □

Proof of Theorem 1.1. Start in L and choose the family from Proposition 4.2. Choose n_*, S_*, S_i and the corresponding ordinals as in Lemma 3.1. In L fix real codes r_i of the ordinals ξ_i, and put $a_i = (n_*, r_i)$. The sequence $\langle a_i : i < \kappa \rangle$ belongs to L.

First force with countable partial functions from κ to 2, ordered by extension, obtaining a new set $A \subseteq \kappa$. This forcing is countably closed. It adds no countable sequences of ordinals, preserves κ, and preserves all the stationary sets S_i, S_*. In $L[A]$ put

$$T = \kappa \setminus \bigcup_{i \in A} S_i.$$

Since $S_* \subseteq T$, the set T is stationary. Force next with $\mathrm{CU}(T)$, and let $C \subseteq T$ be the resulting club. Write $W = L[A][C]$, and let $\mathbb{S}$ be the two-step iteration just described.

By Lemma 5.1, the second step adds no countable sequences of ordinals. The same is therefore true of the two-step iteration. Consequently,

$$\mathbb{R}^W = \mathbb{R}^L, \qquad \omega_1^W = \kappa, \qquad W \models \mathrm{CH}.$$

The properness of our selected family members is now determined exactly by A.

If $i \in A$, then $C \subseteq \kappa \setminus S_i$. By Lemma 5.1, $\mathrm{CU}(\kappa \setminus S_i)$ is proper in W. If $i \notin A$, disjointness gives $S_i \subseteq T$, so Lemma 5.1 preserves its stationarity in W. Therefore $\mathrm{CU}(\kappa \setminus S_i)$ is not proper in W. In either case $\kappa \setminus S_i$ remains stationary, since it contains the preserved stationary set S_*. By Proposition 4.2, the closed presentation still represents this club-shooting forcing. Thus

$$W \models \mathbb{P}_{a_i} \text{ is proper} \iff i \in A. \tag{1}$$

In particular, every one of these selected forcings still adds no countable sequences of ordinals; it is only their properness that varies with the index.

Since W and L have the same reals, $L(\mathbb{R})^W = L$. If the set B^W from the statement belonged to $L(\mathbb{R})^W$, then it would belong to L. The constructible sequence of parameters and (1) would recover $A = \{i < \kappa : a_i \in B^W\}$ in L, contradicting that A is new. This proves $B^W \notin L(\mathbb{R})^W$.

Finally, every projective subset of the reals in W, even one defined using a real parameter, belongs to L. Indeed, all real parameters belong to L, and projective formulas have the same real-quantifier domains and arithmetic interpretation in L and W. Separation in L therefore gives the same set. Thus B^W is not projective. □

Proof of Corollary 1.2. Work in the extension from Theorem 1.1. A jointly closed set can be coded by a tree on tuples of finite sequences. Given a, the tree for its section is obtained by substituting $a \restriction k$ at level k. Membership of any one finite node in this section tree depends on only finitely many digits of a. Thus the map $a \mapsto c(a)$ taking a parameter to closed-set codes for the domain and ordering of $\mathbb{P}_a$ is continuous and has a code in L. Every $c(a)$ is a valid partial-order code.

A projective definition of properness on these codes would have projective preimage B^W, a contradiction. More strongly, if the properness-code set belonged to $L(\mathbb{R})^W = L$, its preimage under the constructible map c would belong to L, again a contradiction. The construction starts in L and is a set-forcing argument, so its relative consistency requires only the consistency of ZFC. □

6. Remarks

The conclusion is a negative consistency answer for presentations whose domain and ordering are analytic, since the counterexamples have these two relations closed. It is not a nonprojectivity theorem for Suslin presentations in the three-relation sense. In the construction the incompatibility graph is coanalytic. Passing to finite independent sets and then to a closed presentation does not establish analyticity of incompatibility. We also make no assertion about separative closed presentations: the auxiliary codes in the presentations constructed here are genuinely redundant.

The argument uses an extension of L with no new reals. It does not establish the conclusion under strong large-cardinal hypotheses. In particular, it does not contradict Zapletal's equivalence, under $\mathsf{ZFC} + \mathrm{CH} + \mathrm{LC}$, between properness and preservation of ω_1 for projective partial orders [5, Lemma 2.1.3]. That additional-hypothesis version and the version requiring analytic incompatibility are not resolved by this construction.

The absence of new reals in the selected forcings is not essential to the nonprojectivity conclusion. The next section proves Theorem 1.3 by appending almost-disjoint coding while retaining closed domain and ordering and obtaining Borel recovery of the generic filter. This additional construction is not used in Theorem 1.1 and Corollary 1.2.

7. A single-real refinement

We now prove Theorem 1.3. The construction applies to the forcing of finite independent sets in any coanalytic graph. We first append almost-disjoint coding, then give an analytic presentation of the iteration with a Borel relation recovering its generic filter. Finally, Lemma 2.3 supplies a closed presentation, and the stationary-set argument preserves the required properness pattern.

7.1. **The generic independent set.** Let E be a coanalytic symmetric irreflexive relation on $X = \omega^\omega$, and let $P(E)$ be the preorder of finite independent sequences defined above. All mentions of X and E as sets in a forcing extension refer to the ground-model sets unless otherwise specified.

If G_0 is $P(E)$-generic, put $H = \bigcup\{\mathrm{supp}(u) : u \in G_0\}$. Then H is independent and

$$G_0 = \{u \in P(E) : \mathrm{supp}(u) \subseteq H\}.$$

For the reverse inclusion, choose for each of the finitely many members of $\mathrm{supp}(u)$ a condition in G_0 containing it, and then a common extension in G_0. Notice that $H \subseteq X^V$, even when the first forcing adds reals.

Fix a bijection between $\omega^{<\omega}$ and ω. For $x \in X$ let

$$a_x = \{\ulcorner x \restriction n \urcorner : n < \omega\}.$$

The map $x \mapsto a_x$ is continuous, each a_x is infinite, and $a_x \cap a_y$ is finite for distinct x, y. Moreover, the union of finitely many sets a_x has infinite complement in ω.

7.2. **The almost-disjoint coding forcing.** Work temporarily in $V[H]$. Let $\mathcal{C}$ be the countable Boolean algebra of clopen subsets of 2^ω, with a fixed discrete coding. For a finite set $F \subseteq H \times \omega$ put

$$Z_F = \{z \in 2^\omega : (\forall (x,k) \in F)(\forall m \in a_x)[m \geq k \implies z(m) = 0]\}.$$

A condition of $\mathbb{A}(H)$ is a pair (C, F), where $C \in \mathcal{C} \setminus \{\varnothing\}$ and $C \cap Z_F$ is nonempty. The ordering is $(C', F') \leq (C, F)$ if $C' \subseteq C$ and $F' \supseteq F$.

The consistency requirement is a finite test. Write C as a finite union of basic cylinders. A cylinder $[s]$ meets Z_F exactly when no position at which s is 1 violates one of the finitely many restraints. If this test holds, extending s by zeros gives a witness. Consequently the consistency requirement is Borel (in fact clopen on the components with fixed finite combinatorial data).

Lemma 7.1. *$\mathbb{A}(H)$ has a dense σ-centered suborder. It adds a real $z \in 2^\omega$ such that, for every $x \in X^V$,*

$$x \in H \iff |\operatorname{supp}(z) \cap a_x| < \omega, \tag{2}$$

where $\operatorname{supp}(z) = \{m : z(m) = 1\}$. *If G_1 is its generic filter, then*

$$G_1 = \{(C, F) \in \mathbb{A}(H) : z \in C \cap Z_F\}. \tag{3}$$

Proof. Conditions whose first coordinate is a basic cylinder $[s]$ are dense. For a fixed s, any finitely many such conditions have a common extension obtained by taking the union of their restraints: none of these restraints forbids a 1 already specified by s, and zeros after s witness joint consistency. Thus these conditions form a σ-centered suborder.

The sets of conditions whose clopen coordinate is contained in a cylinder of prescribed length are dense. The generic filter therefore determines a unique real z, which belongs to every clopen coordinate in the filter and satisfies every restraint occurring there.

For $x \in H$, conditions containing some restraint (x, k) are dense: first refine to a compatible cylinder $[s]$ and then choose $k \geq |s|$. Hence the right side of (2) holds. For $x \in X^V \setminus H$ and $m_0 < \omega$, it is dense to put a 1 at some position in a_x above m_0. Given finitely many restraints, choose this position above the current stem and outside the finitely many sets a_y occurring in the restraints. This is possible by almost disjointness, since none of those y equals x. This proves the converse in (2).

The inclusion from left to right in (3) has already been observed. Conversely, if $z \in C \cap Z_F$, then (C, F) is compatible with every member (D, J) of G_1: the pair $(C \cap D, F \cup J)$ is a condition, with consistency witnessed by z. The finite test for consistency gives the same answer in the ground model of this forcing. A generic filter meets the dense set of conditions either extending a given condition or incompatible with it. Therefore (C, F) belongs to G_1. □

It is important to refine to basic cylinders in the centeredness argument. Conditions with one fixed arbitrary clopen coordinate need not be centered: different restraints can rule out different cylinders in a finite union.

Corollary 7.2. *The extension by $P(E) * \dot{\mathbb{A}}(\dot{H})$ is generated by its coding real z.*

Proof. Equation (2) recovers H from z using the ground-model set X and the fixed family $(a_x : x \in X)$. It then recovers G_0. Equation (3) recovers G_1. Thus $V[G_0][G_1] = V[z]$. □

The coding real is new even over $V[H]$. Given a condition and any real $z_0 \in V[H]$, refine to a compatible cylinder $[s]$. Choose a position above $|s|$ outside the finitely many restrained branches and set its value opposite to z_0 at that position. This gives an extension forcing $z \neq z_0$.

7.3. **An analytic presentation and a Borel recovery relation.** To control definability, use the following presentation of the iteration directly in V. A condition of $R(E)$ is a triple $p = (u, C, F)$, where $u \in P(E)$, $C \in \mathcal{C} \setminus \{\varnothing\}$, and $F \subseteq \operatorname{supp}(u) \times \omega$ is finite with $C \cap Z_F \neq \varnothing$. Finite sets are represented by finite sequences, and inclusion always means inclusion of their supports. Put

$$(v, D, J) \leq (u, C, F) \iff \operatorname{supp}(v) \supseteq \operatorname{supp}(u), \quad D \subseteq C, \quad J \supseteq F.$$

The domain is analytic: independence of u is analytic, incidence of the finite restraints with $\operatorname{supp}(u)$ is closed, and consistency of the restraints with C is Borel. The ordering is the restriction of a closed preorder on the ambient space of all triples, since finite support inclusion is closed and the clopen codes are discrete. These assertions hold uniformly for a coanalytic family of graphs E.

Lemma 7.3. *$R(E)$ is a dense presentation of $P(E) * \dot{\mathbb{A}}(\dot{H})$.*

Proof. Send (u, C, F) to $(u, (\check{C}, \check{F}))$. The first coordinate forces all vertices occurring in F to belong to $\dot{H}$, so this is an iteration condition. The map preserves and reflects order on these conditions.

Given an arbitrary iteration condition, strengthen its first coordinate to decide the finite data of the second coordinate. This is possible because its clopen code, integer thresholds, and vertices all belong to ground-model sets, the vertices being in X^V. Suppose the second coordinate is decided as (C, F). The first coordinate now forces all the vertices of F into $\dot{H}$. Adjoining them explicitly to its finite independent sequence gives a condition: an incompatibility among these vertices or with the old sequence would contradict that forced statement. We have obtained a stronger condition in the displayed image. Thus the image is dense; in particular compatibility is also preserved and reflected. □

For a raw triple $p = (u, C, F)$ and $z \in 2^\omega$ define

$$\chi_0(p, z) \iff \begin{aligned} & z \in C, \\ & (\forall x \in \operatorname{supp}(u))\ |\operatorname{supp}(z) \cap a_x| < \omega, \\ & (\forall (x, k) \in F)\ \operatorname{supp}(z) \cap a_x \subseteq k. \end{aligned} \tag{4}$$

This is Borel, uniformly on the ambient space of triples and reals. For example the middle clause says, for each of finitely many x, that there is a natural-number bound after which z is zero at all positions in a_x.

Proposition 7.4. *If G is $R(E)$-generic and z is the corresponding coding real, then*

$$G = \{p \in R(E) : \chi_0(p, z)\}.$$

For each nonempty proper clopen set O, the two conditions $(\varnothing, O, \varnothing)$ and $(\varnothing, 2^\omega \setminus O, \varnothing)$ form a maximal antichain, and their recovery relations are exactly membership in O and in its complement.

Proof. A condition in G satisfies (4) by Lemmas 7.1 and 7.3. Conversely, let $p = (u, C, F)$ satisfy (4). Equation (2) gives $\operatorname{supp}(u) \subseteq H$. For any $q = (v, D, J) \in G$, the union of the supports of u and v is independent because it is contained in H. The clopen coordinate $C \cap D$ and restraint set $F \cup J$ are consistent, as witnessed by z. The finite consistency test therefore gives a ground-model condition extending both p and q. Thus p is compatible with every condition in G, which implies $p \in G$ by genericity.

The two displayed conditions are incompatible because their clopen coordinates are disjoint. For any condition, a real witnessing its consistency belongs to one of the two clopen sets; intersecting with that clopen set gives a strengthening compatible with the corresponding displayed condition. The antichain is therefore maximal. The two formulas in (4) reduce to clopen membership because the finite sequences and restraints are empty. □

Proposition 7.5. *The forcing $R(E)$ has a closed presentation whose generic filter is recovered by a Borel relation and which retains the antichains from Proposition 7.4. The construction is uniform for coanalytic families of graphs.*

Proof. Apply Lemma 2.3 to the analytic domain of $R(E)$ and its closed ambient preorder. Explicitly, choose a closed projection representation for that domain, and use conditions (p, w, k), where w is a witness and $k < \omega$. The ordering is equality or the conjunction of the old ordering with strict increase of k.

Projection to p preserves compatibility and has the extension lifting property. It induces an isomorphism of separative quotients. If $\widehat{G}$ is generic, its projection generates the old generic filter G. Every lift of every member of G belongs to $\widehat{G}$: it is compatible with every member of $\widehat{G}$, and genericity applies. Consequently $\chi((p, w, k), z) = \chi_0(p, z)$ recovers $\widehat{G}$ exactly and is Borel.

Arbitrary lifts of the two old antichain members still form a maximal antichain, by preservation of compatibility and the lifting property.

The real itself also has a Borel name in this presentation. For its mth bit, take all conditions whose clopen coordinate is contained in $\{z : z(m) = 1\}$. The dense sets deciding a long enough initial segment show that this name evaluates to the coding real. $\square$

7.4. **Preserving the properness pattern.** Appending an arbitrary forcing would not in general preserve nonproperness. The construction here uses a specific witness to nonproperness that persists through the second step.

Lemma 7.6. *If $P(E)$ is proper, then $R(E)$ is proper. If $P(E)$ preserves ω_1 and forces a ground-model stationary set $S \subseteq \omega_1$ to be nonstationary by adding a club disjoint from it, then $R(E)$ is not proper.*

Proof. The second iterand has a dense σ-centered suborder by Lemma 7.1. It is therefore ccc and proper. The two-step iteration theorem for proper forcing gives the first assertion [1].

For the second assertion, the second iterand preserves ω_1, and the first iterand's club remains closed and unbounded. The iteration destroys the stationarity of S, so it is not proper. Use Lemma 7.3 to transfer both conclusions to $R(E)$. $\square$

Proof of Theorem 1.3. Use the coanalytic graphs E_a from Lemma 4.1 and the extension W constructed in the proof of Theorem 1.1, which adds no countable sequences of ordinals over L. In the notation of that proof, $\kappa = \omega_1^L = \omega_1^W$, the sequence $\langle a_i : i < \kappa \rangle$ belongs to L, and

$$P(E_{a_i}) \simeq \mathrm{CU}(\kappa \setminus S_i).$$

A new set $A \subseteq \kappa$ has been arranged so that S_i is nonstationary for $i \in A$ and remains stationary for $i \notin A$. Each complement $\kappa \setminus S_i$ remains stationary. If $i \in A$, the club-shooting forcing is proper. If $i \notin A$, it preserves κ but destroys the stationarity of S_i.

Form $R(E_a)$ uniformly and replace it by its closed presentation $\mathbb{R}_a$ using Proposition 7.5. These constructions have codes in L. By Lemma 7.6,

$$W \models \mathbb{R}_{a_i} \text{ is proper} \quad \iff \quad i \in A.$$

All the generating-real and recovery assertions follow from Corollary 7.2 and Propositions 7.4 and 7.5.

Finally $L(\mathbb{R})^W = L$. If the properness-parameter set for this new closed family belonged to $L(\mathbb{R})^W$, it would recover A in L from the constructible sequence of parameters, a contradiction. Every projective set with real parameters belongs to $L(\mathbb{R})^W$, so the same properness-parameter set is not projective. $\square$

The coanalytic graphs, closed presentations, and coding relation remain the same in W as in L, because there are no new reals. This fact concerns the outer consistency construction. It does not say that the individual forcings $\mathbb{R}_a$, when subsequently used over W, add no reals: they do add their coding reals, as required.

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