THE DESCRIPTIVE COMPLEXITY OF ROOTS OF MEASURE-PRESERVING TRANSFORMATIONS

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Abstract. For an integer $n \geq 2$, let $\mathcal{R}_n$ be the set of invertible measure-preserving transformations that admit an nth root. We prove that $\mathcal{R}_n$ is $\mathbf{\Sigma}^1_1$-complete for every $n \geq 2$. More generally, for every nonempty $A \subseteq \{2, 3, \ldots\}$, the transformations admitting an nth root for some $n \in A$ form a $\mathbf{\Sigma}^1_1$-complete set. The case $n = 2$ proves the conjecture in Problem 43 of the Tianyuan Workshop problem list. The proof starts from the continuous reduction of Foreman, Rudolph, and Weiss from ill-founded trees to transformations conjugate to their inverses. Their analysis of graph joinings also implies that every transformation in the reducing family has no mth root for any $m \geq 2$. A finite direct-sum construction then converts conjugacy to the inverse into the existence of an nth root.

1. Introduction

Let

$$G = \mathrm{Aut}([0,1], \lambda)$$

be the group of invertible measure-preserving transformations modulo equality almost everywhere, equipped with the weak topology. Thus $T_k \to T$ if and only if

$$\lambda(T_k A \triangle TA) \longrightarrow 0$$

for every measurable $A \subseteq [0,1]$. Throughout, $\cong$ denotes measure-theoretic conjugacy. For $n \geq 2$, put

$$\mathcal{R}_n = \{T \in G : (\exists S \in G)\ S^n = T\}.$$

We use *complete analytic* to mean $\mathbf{\Sigma}^1_1$-complete under continuous reducibility. Since the power map $S \mapsto S^n$ is continuous, $\mathcal{R}_n$ is analytic.

The existence of square roots of measure-preserving transformations was studied by Halmos in [3]; see also the question on [4, p. 97]. Problem 43 of the 2025 Tianyuan Workshop problem list asks for a characterization of $\mathcal{R}_2$ and records the conjecture that this set is complete analytic [1].

Note that the conjecture does not require the transformations be ergodic, which allows us to use a simple trick in proving the following stronger statement.

Theorem 1. *For every integer $n \geq 2$, the set $\mathcal{R}_n$ is $\mathbf{\Sigma}^1_1$-complete. The continuous reduction witnessing hardness takes values among transformations with exactly n ergodic components, each of measure $1/n$.*

The more interesting question, which asks if the square roots of *ergodic* measure-preserving transformations are complete analytic, remains open.

Foreman, Rudolph, and Weiss constructed in [2] a continuous family of ergodic transformations satisfying

$$\tau \in \mathsf{IF} \quad \Longleftrightarrow \quad R_\tau \cong R_\tau^{-1},$$

where IF is the complete analytic set of ill-founded trees. We first extract from their analysis of graph joinings an additional property of this family: no R_τ admits an mth root for any $m \geq 2$. We then use the finite direct sum

$$D_n(R) = \underbrace{R \oplus \cdots \oplus R}_{n-1 \text{ copies}} \oplus R^{-1}.$$

For the Foreman–Rudolph–Weiss family, $D_n(R)$ has an nth root exactly when R is conjugate to R^{-1}.

2. The Foreman–Rudolph–Weiss family

Let Tr be the Polish space of trees on $\mathbb{N}$ of unbounded height used in [2], and let $\mathsf{IF} \subseteq \mathsf{Tr}$ be the set of ill-founded trees. The set IF is $\boldsymbol{\Sigma}^1_1$-complete [2, Theorem 4].

Let $\tau \mapsto \nu_\tau$ be the continuous map of [2, Theorem 7], and let K_τ be the support of ν_τ. Then $(K_\tau, \nu_\tau, \mathrm{sh})$ is ergodic and

$$\tau \in \mathsf{IF} \iff (K_\tau, \nu_\tau, \mathrm{sh}) \cong (K_\tau, \nu_\tau, \mathrm{sh}^{-1}). \tag{1}$$

The interval realization following [2, Corollary 9] gives a continuous map

$$\tau \longmapsto R_\tau \in G$$

such that R_τ is measure-theoretically isomorphic to $(K_\tau, \nu_\tau, \mathrm{sh})$ for every τ. We prove the absence of roots in the symbolic model and then transfer it to R_τ by conjugacy.

Fix $\tau \in \mathsf{Tr}$. The symbolic system has an odometer factor

$$\pi_\tau : K_\tau \longrightarrow \mathcal{O}_\tau$$

for which

$$\pi_\tau(\mathrm{sh}\, x) = \pi_\tau(x) + \overline{1} \tag{2}$$

for almost every x; see [2, Section 5.1 and Lemma 22]. The coordinate moduli of $\mathcal{O}_\tau$ form an infinite sequence $q_0, q_1, \ldots$, with $q_i \geq 2$ for every i. For $j \in \mathbb{Z}$, write $\overline{j} = j\overline{1}$.

Lemma 2. *If $Q \in \mathrm{Aut}(K_\tau, \nu_\tau)$ commutes with* sh*, then there is an integer j such that*

$$\pi_\tau(Qx) = \pi_\tau(x) + \overline{j}$$

for almost every x.

Proof. Let $\Delta_Q = (\mathrm{id}, Q)_* \nu_\tau$ be the graph joining of Q. Corollary 40 of [2] gives an integer j and an even element g of the inverse-limit group $G_\infty(\tau)$ such that $\eta_g \circ (\mathrm{id}, \mathrm{sh}^j)$ is supported on the graph of Q. Since this joining has first marginal ν_τ, it is equal to Δ_Q.

For each finite factor, Lemma 27 of [2] says that the graph joining determined by an even element projects to the diagonal on the zeroth factor. The zeroth factor is the odometer $\mathcal{O}_\tau$; see [2, Section 5.2]. Lemma 26 and the paragraph following Lemma 27 show that the inverse-limit graph joining η_g has these finite-factor projections. Its projection to $\mathcal{O}_\tau \times \mathcal{O}_\tau$ is therefore concentrated on the diagonal. Applying sh^j in the second coordinate changes this projection to the graph of the translation $z \mapsto z + \overline{j}$. The conclusion follows. $\square$

Proposition 3. *For every $\tau \in \mathsf{Tr}$ and every integer $m \geq 2$, the transformation R_τ has no mth root.*

Proof. Root existence is invariant under measure-theoretic conjugacy, so it is enough to work with $(K_\tau, \nu_\tau, \mathrm{sh})$. Suppose that $Q^m = \mathrm{sh}$ for some $m \geq 2$. Then Q commutes with sh. By Lemma 2, there is $j \in \mathbb{Z}$ such that

$$\pi_\tau(Qx) = \pi_\tau(x) + \overline{j}$$

for almost every x. Iterating and using (2), we get

$$\pi_\tau(x) + \overline{mj} = \pi_\tau(Q^m x) = \pi_\tau(\mathrm{sh}\, x) = \pi_\tau(x) + \overline{1}.$$

Hence $(mj-1)\overline{1} = 0$ in $\mathcal{O}_\tau$.

The element $\overline{1}$ has infinite order. Indeed, its image in the quotient determined by the first $r+1$ odometer coordinates has order $q_0 q_1 \cdots q_r$, and these orders are unbounded. It follows that $mj = 1$, which is impossible for $m \geq 2$ and $j \in \mathbb{Z}$. $\square$

We shall use the following consequences of the Foreman–Rudolph–Weiss construction:

$$\begin{aligned} &\tau \longmapsto R_\tau \text{ is continuous, and every } R_\tau \text{ is ergodic;} \\ &\tau \in \mathsf{IF} \iff R_\tau \cong R_\tau^{-1}; \\ &\text{no } R_\tau \text{ has an } m\text{th root for any } m \geq 2. \end{aligned} \tag{3}$$

3. Finite direct sums

For $n \geq 2$, let

$$X_n = [0,1] \times \{0, 1, \dots, n-1\}, \qquad \lambda_n = \lambda \times u_n,$$

where u_n is the uniform probability measure. Fix a measure-space isomorphism from (X_n, λ_n) to $([0,1], \lambda)$. For $T_0, \dots, T_{n-1} \in G$, define

$$(T_0 \oplus \cdots \oplus T_{n-1})(x, i) = (T_i x, i)$$

on X_n and use the fixed isomorphism to regard this direct sum as an element of G.

Lemma 4. *For each $n \geq 2$, the map*

$$G^n \longrightarrow G, \qquad (T_0, \dots, T_{n-1}) \longmapsto T_0 \oplus \cdots \oplus T_{n-1},$$

is continuous. Consequently, so is the map

$$D_n : G \longrightarrow G, \qquad D_n(R) = \underbrace{R \oplus \cdots \oplus R}_{n-1 \text{ copies}} \oplus R^{-1}.$$

Proof. It is enough to work on (X_n, λ_n), since conjugation by the fixed measure-space isomorphism is a homeomorphism of the corresponding automorphism groups. Suppose that $T_i^{(k)} \to T_i$ for every $i < n$. For a measurable set $A \subseteq X_n$, put

$$A_i = \{x \in [0,1] : (x, i) \in A\}.$$

Then

$$\begin{aligned} \lambda_n\left(\left(\bigoplus_{i<n} T_i^{(k)}\right) A \mathbin{\triangle} \left(\bigoplus_{i<n} T_i\right) A\right) \\ = \frac{1}{n} \sum_{i<n} \lambda\big(T_i^{(k)} A_i \mathbin{\triangle} T_i A_i\big) \longrightarrow 0. \end{aligned}$$

This is convergence in the weak topology. The last assertion follows from the continuity of inversion in G. $\square$

Proposition 5. *Let $R \in G$ be ergodic and suppose that R has no mth root for any $m \geq 2$. For every $n \geq 2$,*

$$D_n(R) \in \mathcal{R}_n \quad \Longleftrightarrow \quad R \cong R^{-1}.$$

Moreover, if $R \ncong R^{-1}$, then $D_n(R)$ has no mth root for any $m \geq 2$.

Proof. Suppose first that $R \cong R^{-1}$. Then $D_n(R)$ is conjugate to the direct sum $R^{\oplus n}$ of n copies of R. On X_n, define

$$C(x,i) = \begin{cases} (x, i+1), & i < n-1, \\ (Rx, 0), & i = n-1. \end{cases}$$

Then C is invertible and measure preserving, and $C^n = R^{\oplus n}$. Thus $D_n(R)$ has an nth root.

Conversely, suppose that $R \ncong R^{-1}$ and that $S^m = D_n(R)$ for some $m \geq 2$. Then S commutes with $D_n(R)$. Let

$$E_i = [0,1] \times \{i\}, \qquad i < n.$$

Since the restriction of $D_n(R)$ to each E_i is ergodic, the sets $E_0, \ldots, E_{n-1}$ are precisely the atoms of its invariant σ-algebra. Commutation with $D_n(R)$ implies that S permutes these atoms modulo null sets. Moreover, the restriction of S to any atom is a conjugacy between the corresponding restrictions of $D_n(R)$.

The atom E_{n-1} is the unique one on which the restriction of $D_n(R)$ is isomorphic to R^{-1}; on every other atom it is isomorphic to R. Since $R \ncong R^{-1}$, the atom E_{n-1} must therefore be fixed by S modulo null sets. Restricting $S^m = D_n(R)$ to this atom gives an automorphism Q with $Q^m = R^{-1}$. Hence $(Q^{-1})^m = R$, contrary to the hypothesis on R. □

4. Proof of the main theorem

Proof of Theorem 1. Fix $n \geq 2$. The set $\mathcal{R}_n$ is analytic because it is the range of the continuous map $S \mapsto S^n$ on the Polish space G.

Define

$$\Phi_n : \mathsf{Tr} \longrightarrow G, \qquad \Phi_n(\tau) = D_n(R_\tau).$$

This map is continuous by Lemma 4. By Proposition 5 and (3),

$$\tau \in \mathsf{IF} \quad \Longleftrightarrow \quad R_\tau \cong R_\tau^{-1} \quad \Longleftrightarrow \quad \Phi_n(\tau) \in \mathcal{R}_n.$$

Thus Φ_n continuously reduces the complete analytic set IF to $\mathcal{R}_n$, so $\mathcal{R}_n$ is $\mathbf{\Sigma}^1_1$-complete.

Every R_τ is ergodic. Hence $\Phi_n(\tau)$ has exactly n ergodic components, namely its coordinate components, and each has measure $1/n$. □

The case $n = 2$ proves the conjecture in Problem 43.

Corollary 6. *The set*

$$\{T \in \mathrm{Aut}([0,1], \lambda) : (\exists S \in \mathrm{Aut}([0,1], \lambda))\ S^2 = T\}$$

is $\mathbf{\Sigma}^1_1$-complete.

The last assertion of Proposition 5 also gives the following simultaneous extension.

Corollary 7. *Let A be any nonempty subset of $\{2, 3, \ldots\}$. Then*

$$\mathcal{R}_A = \{T \in G : (\exists n \in A)(\exists S \in G)\ S^n = T\} = \bigcup_{n \in A} \mathcal{R}_n$$

is $\mathbf{\Sigma}^1_1$-complete. In particular, the set of transformations admitting an mth root for at least one $m \geq 2$ is $\mathbf{\Sigma}^1_1$-complete.

Proof. The set $\mathcal{R}_A$ is analytic as a countable union of analytic sets. Choose $n \in A$ and use the map Φ_n from the proof of Theorem 1. If $\tau \in \mathsf{IF}$, then $\Phi_n(\tau)$ has an nth root. If $\tau \notin \mathsf{IF}$, then $R_\tau \ncong R_\tau^{-1}$, so Proposition 5 shows that $\Phi_n(\tau)$ has no mth root for any $m \geq 2$. Therefore

$$\tau \in \mathsf{IF} \quad \Longleftrightarrow \quad \Phi_n(\tau) \in \mathcal{R}_A.$$

□

References

[1] G. Barmpalias, N. Bazhenov, C. T. Chong, W. Dai, S. Gao, J. L. Goh, J. He, K. M. S. Ng, A. Nies, T. Slaman, R. Thornton, W. Wang, J. Yu, and L. Yu, *Open problems in computability theory and descriptive set theory*, arXiv:2507.03972 [math.LO], 2025.

[2] M. Foreman, D. J. Rudolph, and B. Weiss, *The conjugacy problem in ergodic theory*, Ann. of Math. (2) **173** (2011), no. 3, 1529–1586.

[3] P. R. Halmos, *Square roots of measure preserving transformations*, Amer. J. Math. **64** (1942), 153–166.

[4] P. R. Halmos, *Lectures on ergodic theory*, Publications of the Mathematical Society of Japan, vol. 3, Mathematical Society of Japan, Tokyo, 1956.

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