Second-Order Constructibility and HOD Without AC

An exposition on Szczepaniak's construction

Based on Zbigniew Szczepaniak, 1977

Abstract

Myhill and Scott proved in ZFC that the hierarchy obtained by iterating full second-order definability is exactly HOD. Szczepaniak showed that Choice cannot simply be deleted from this theorem: assuming ZF is consistent, so is

$$\mathsf{ZF} + L^{(2)} \neq \mathrm{HOD}.$$

The original proof occupies only six pages and uses notation and conventions that are now unfamiliar. These notes rewrite the argument in modern language, make the symmetric-extension construction explicit, and expand the two most compressed parts of the paper: the closed-tail localization lemma and the recovery of the auxiliary sequence Z from the sets of ordinals. In fact the constructed model satisfies

$$L^{(2)} \subsetneq \mathrm{HOD}.$$

The intended reader knows ordinary forcing, the basic theory of symmetric models, and examples such as Cohen's first and second models. We also include a proof of the Myhill–Scott theorem under Choice, so that the positive and negative results can be read side by side.

Notation and source. Szczepaniak writes the second-order constructible universe as L^1, uses ω_α for the initial ordinal $\aleph_\alpha$, and denotes his forcing at κ by $P(\kappa)$. We write $L^{(2)}$, $\aleph_\alpha$, and $\mathbb{Q}(\kappa)$, respectively. We retain the standard notation $N[G/\mathcal{F}]$ for a symmetric submodel, and we write $L(\mathcal{P}(\mathrm{On}))$ for the inner model generated by all sets of ordinals (Szczepaniak's L^*). The paper suppresses several standard intermediate-model and homogeneity arguments. They are isolated below as separate lemmas so that the logical role of each is visible. The proof of the ZFC equality $L^{(2)} = \mathrm{HOD}$ in Section 3 is not part of Szczepaniak's six-page argument; it is included for context from the Myhill–Scott theorem he cites at the outset.

Contents

1 What the theorem says

Let $L^{(2)}$ be the class obtained by starting with the empty set and, at each successor stage, taking all subsets of the previous stage that are definable by *full second-order* formulas with parameters from that stage. Myhill and Scott proved

$$\mathsf{ZFC} \vdash L^{(2)} = \mathrm{HOD}.$$

Szczepaniak's theorem is the following relative consistency result.

Theorem 1.1 (Szczepaniak)**.** *If* ZF *is consistent, then so is*

$$\mathsf{ZF} + L^{(2)} \neq \mathrm{HOD}.$$

Indeed, the countermodel constructed below satisfies $L^{(2)} \subsetneq \mathrm{HOD}$*.*

Szczepaniak states the conclusion as an inequality. The strict-inclusion formulation is the immediate consequence of the ZF inclusion proved in Theorem 2.2 below.

Proof roadmap

The proof separates two kinds of information.

1. The class $L^{(2)}$ depends only on the sets of ordinals in the ambient universe.
2. We construct two transitive models

$$N_1 \subseteq N_2$$

with exactly the same sets of ordinals.
3. There is a Cohen real $a \in N_1$ such that

$$a \notin \mathrm{HOD}^{N_1} \qquad \text{but} \qquad a \in \mathrm{HOD}^{N_2}.$$

4. Since N_1 and N_2 have the same $L^{(2)}$, while ZF proves $L^{(2)} \subseteq \mathrm{HOD}$, the real a belongs to $\mathrm{HOD}^{N_2} \setminus (L^{(2)})^{N_2}$.

The forcing creates non-well-orderable families S_n of generic subsets of suitably spaced regular cardinals high in the L-cardinal sequence. An extra object B records exactly those S_n whose indices belong to a. The object B adds no sets of ordinals, but it makes the bits of a ordinal definable. That is the mechanism by which HOD changes while $L^{(2)}$ does not.

2 The second-order constructible hierarchy

2.1 Full second-order definability

For a set A, let $\mathrm{Def}_2(A)$ be the family of all $X \subseteq A$ for which there are a full second-order formula φ and first-order parameters $\bar{a} \in A^{<\omega}$ such that

$$X = \{x \in A : (A, \in) \models_2 \varphi(x, \bar{a})\}.$$

The notation $\models_2$ denotes *full* semantics: a second-order variable ranges over every relation of the appropriate finite arity on A that exists in the ambient universe. This is not Henkin semantics.

Definition 2.1. Define

$$L_0^{(2)} = \varnothing, \qquad L_{\alpha+1}^{(2)} = \mathrm{Def}_2(L_\alpha^{(2)}),$$

and, for limit λ,

$$L_\lambda^{(2)} = \bigcup_{\alpha<\lambda} L_\alpha^{(2)}.$$

Finally,

$$L^{(2)} = \bigcup_{\alpha\in\mathrm{Ord}} L_\alpha^{(2)}.$$

The operation $A \mapsto \mathrm{Def}_2(A)$ is not generally absolute between inner models. Two models can have the same set A but different collections of relations on A, and hence different full second-order truth predicates for $(A, \in)$.

2.2 The inclusion $L^{(2)} \subseteq \mathrm{HOD}$ needs no Choice

Proposition 2.2. *ZF proves*

$$L^{(2)} \subseteq \mathrm{HOD}.$$

Proof. Proceed by induction on α. The level $L_\alpha^{(2)}$ is definable from α. Full second-order satisfaction for a set-sized structure is itself first-order definable in the ambient universe: second-order quantifiers are implemented by quantification over subsets and finite-arity relations on the domain.

Suppose every member of $L_\alpha^{(2)}$ is hereditarily ordinal definable and $X \in L_{\alpha+1}^{(2)}$. Choose a second-order definition of X over $L_\alpha^{(2)}$ from parameters $a_0, \ldots, a_{k-1} \in L_\alpha^{(2)}$. Replace each a_i by an ordinal definition of it. The resulting description defines X from ordinals alone. Moreover $X \subseteq L_\alpha^{(2)}$, so every member of the transitive closure of X is hereditarily ordinal definable by the induction hypothesis. Thus $X \in \mathrm{HOD}$. Limit stages are immediate. □

Consequently, any failure of equality must have the form

$$L^{(2)} \subsetneq \mathrm{HOD}.$$

3 The Myhill–Scott theorem under Choice

Szczepaniak begins his paper by citing the theorem of Myhill and Scott [3] that ZFC proves

$$L^{(2)} = \mathrm{HOD}.$$

We have already proved the inclusion from left to right, and that argument did not use Choice. The reverse inclusion is where Choice enters. The basic idea is memorable: an ordinal definition can be made inside some rank segment V_α, and full second-order logic can quantify over a relation coding a copy of that V_α. Choice is what allows us to fit such a copy inside a sufficiently large stage of $L^{(2)}$.

3.1 How second-order logic recognizes a rank segment

We first isolate the model-theoretic observation behind the proof. We use only limit ordinals below; this is harmless because the reflection theorem supplies arbitrarily large limit ordinals with the desired reflection properties.

Lemma 3.1 (Recognizing V_α in full second-order logic)**.** *There is a single full second-order formula*

$$\mathrm{VCopy}(M, E, \alpha)$$

with a first-order parameter α and second-order variables M and E such that, whenever D is a transitive set containing α and α is a limit ordinal,

$$(D, \in) \models_2 \mathrm{VCopy}(M, E, \alpha)$$

if and only if $M \subseteq D$, $E \subseteq M \times M$, and

$$(M, E) \cong (V_\alpha, \in).$$

Explanation. One convenient formulation asks that E be well-founded and extensional and then uses a second-order relation to assign to every member of M its E-rank, an ordinal below α. Finally one imposes the *fullness* condition

every E-rank-bounded subset of M is the E-predecessor set of some member of M.

This condition is second order: the phrase "every subset" is expressed by quantifying over a unary second-order variable $X \subseteq M$.

By well-founded extensionality, (M, E) has a Mostowski collapse to a transitive set T. The rank clause gives $T \subseteq V_\alpha$. Conversely, fullness shows by induction on $\xi < \alpha$ that $V_\xi \subseteq T$: once all sets of rank below ξ occur, every subset of the earlier level occurs as the predecessor set of another element. Hence $T = V_\alpha$. Conversely, $(V_\alpha, \in)$ clearly satisfies these conditions.

All of this is expressible by one second-order formula. For example, well-foundedness of E is expressed by saying that every nonempty $X \subseteq M$ has an E-minimal member. The formula is uniform in the first-order parameter α. $\square$

3.2 Putting a copy inside a stage of $L^{(2)}$

The following elementary use of Choice is the step that fails in the choiceless setting.

Lemma 3.2 (Anchored copies)**.** *Assume AC. Let α be an ordinal, let D be a set with*

$$|D| > |V_\alpha|,$$

and let $T \subseteq D \cap V_\alpha$. Then there are $M \subseteq D$ and $E \subseteq M^2$ together with an isomorphism

$$\pi : (V_\alpha, \in) \longrightarrow (M, E)$$

which fixes every member of T.

Proof. Since $|T| \leq |V_\alpha| < |D|$, Choice gives

$$|D \setminus T| = |D| > |V_\alpha \setminus T|.$$

Choose an injection of $V_\alpha \setminus T$ into $D \setminus T$, extend it by the identity on T, and call the resulting injection π. Let $M = \pi``V_\alpha$, and transport membership along π:

$$\pi(x)\ E\ \pi(y) \quad \Longleftrightarrow \quad x \in y.$$

□

Notice that every stage $L^{(2)}_\theta$ is canonically well-orderable even in ZF: definitions can be ordered by formula code and then by the previously constructed well-order of the parameters. Under AC, every V_α is also well-orderable, so by taking θ large enough we can ensure

$$|L^{(2)}_\theta| > |V_\alpha|.$$

Without AC, a set such as V_α may fail to be well-orderable, and then it cannot be injected into *any* of the well-orderable stages $L^{(2)}_\theta$. This is the obstruction that Szczepaniak's model exploits.

3.3 The Myhill–Scott proof

Theorem 3.3 (Myhill–Scott)**.** *ZFC proves*

$$L^{(2)} = \mathrm{HOD}.$$

Proof. By Theorem 2.2, it remains to prove $\mathrm{HOD} \subseteq L^{(2)}$. We argue by induction on rank. Let $x \in \mathrm{HOD}$ and assume that every member of x already belongs to $L^{(2)}$. Since x is a set, there is a stage θ_0 such that

$$x \subseteq L^{(2)}_{\theta_0}.$$

Because x is ordinal definable, there are a first-order formula $\varphi(u, \vec{\beta})$ and a finite tuple of ordinal parameters $\vec{\beta}$ such that

$$u \in x \quad \Longleftrightarrow \quad V \models \varphi(u, \vec{\beta}). \tag{1}$$

By Lévy reflection, choose a limit ordinal α above the ranks of x and the parameters such that, for every $u \in V_\alpha$,

$$V \models \varphi(u, \vec{\beta}) \quad \Longleftrightarrow \quad V_\alpha \models \varphi(u, \vec{\beta}). \tag{2}$$

Choose $\theta \geq \theta_0$ so large that

$$\alpha, \vec{\beta} \in L^{(2)}_\theta \qquad \text{and} \qquad |L^{(2)}_\theta| > |V_\alpha|.$$

This is possible under AC; for instance, take θ above a cardinal larger than $|V_\alpha|$. Put $D = L^{(2)}_\theta$.

We now define x over D by a full second-order formula. For $u \in D$, let $\Phi(u, \alpha, \vec{\beta})$ assert that there are a unary second-order relation $M \subseteq D$, a binary relation $E \subseteq M^2$, and a unary relation $T \subseteq D$ such that:

1. T is the transitive closure, with respect to the actual membership relation, of $\{u\} \cup \{\beta_i : i < |\vec{\beta}|\}$;
2. $T \subseteq M$ and E agrees with actual membership on T;
3. $\mathrm{VCopy}(M, E, \alpha)$ holds;
4. the formula $\varphi(u, \vec{\beta})$ holds in the structure (M, E), i.e. its first-order quantifiers are relativized to M and every occurrence of $\in$ is replaced by E.

This is one second-order formula over D. In particular, M, E, and T are quantified second-order objects rather than parameters.

We claim that for every $u \in D$,

$$u \in x \quad \Longleftrightarrow \quad (D, \in) \models_2 \Phi(u, \alpha, \vec{\beta}). \tag{3}$$

Suppose first that $u \in x$. Then $u \in V_\alpha$ by our choice of α. Let T be the indicated transitive closure. Since D is transitive and contains u and the ordinal parameters, $T \subseteq D$; also $T \subseteq V_\alpha$. By Theorem 3.2, there is a copy (M, E) of $(V_\alpha, \in)$ inside D whose isomorphism fixes T. By Equation (2), this copy satisfies the relativization of $\varphi(u, \vec{\beta})$, so Φ holds.

Conversely, suppose $\Phi(u, \alpha, \vec{\beta})$ holds. By Theorem 3.1, the Mostowski collapse of (M, E) is $(V_\alpha, \in)$. Because T is transitive and E agrees with actual membership on T, the collapse fixes every member of T pointwise. In particular it fixes u and each β_i. Hence

$$V_\alpha \models \varphi(u, \vec{\beta}),$$

and Equations (1) and (2) gives $u \in x$. This proves Equation (3).

Thus x is a subset of $D = L^{(2)}_\theta$ definable over D by a full second-order formula with the permitted first-order parameters $\alpha, \vec{\beta} \in D$. Therefore

$$x \in \mathrm{Def}_2(L^{(2)}_\theta) = L^{(2)}_{\theta+1}.$$

The rank induction is complete. □

Remark 3.4 (Exactly where Choice is used). The logical part of the argument — recognizing a well-founded extensional copy of V_α and evaluating φ inside it — uses full second-order semantics but not Choice. AC is used to make V_α well-orderable and thereby place an isomorphic copy of it inside a sufficiently large, well-orderable stage of $L^{(2)}$. This pinpoints what one must defeat in a choiceless counterexample.

4 The invariant behind Szczepaniak's argument

4.1 The inner model generated by all sets of ordinals

For a universe $V \models \mathsf{ZF}$, put

$$\mathcal{P}(\mathrm{On})^V = \bigcup_{\theta \in \mathrm{Ord}^V} \mathcal{P}^V(\theta).$$

Following the usual notation, write

$$(L(\mathcal{P}(\mathrm{On})))^V = \bigcup_{\theta \in \mathrm{Ord}^V} L^V(\mathcal{P}^V(\theta)).$$

When the ambient universe is clear, we simply write $L(\mathcal{P}(\mathrm{On}))$. This is Szczepaniak's L^*. It contains every set of ordinals of the ambient universe.

Lemma 4.1 (Szczepaniak's key observation)**.** *For every $V \models \mathsf{ZF}$,*

$$(L^{(2)})^V = (L^{(2)})^{(L(\mathcal{P}(\mathrm{On})))^V}.$$

Proof. Let $K = (L(\mathcal{P}(\mathrm{On})))^V$. We prove by induction on α that

$$(L^{(2)}_\alpha)^V = (L^{(2)}_\alpha)^K.$$

At the same time, the common level is well-orderable in K. This is clear at stage 0, and at a successor stage one can order definitions first by the code of the formula and then by the previously constructed well-order of the finite parameter tuples.

Assume the levels agree at α, and write their common value as A. Choose in K an injection

$$j : A \longrightarrow \mathrm{Ord}.$$

If $X \subseteq A$ belongs to V, then $j``X$ is a set of ordinals of V and therefore belongs to K. Since $j \in K$,

$$X = \{x \in A : j(x) \in j``X\}$$

also belongs to K. Hence

$$\mathcal{P}^V(A) = \mathcal{P}^K(A).$$

The same argument, after coding tuples, gives equality of all finite-arity relations on A. Therefore V and K have the same full second-order semantics for $(A, \in)$ and produce the same successor level. The limit step is immediate. □

Corollary 4.2. *If U and V are transitive models of ZF with the same ordinals and the same sets of ordinals, then*

$$(L^{(2)})^U = (L^{(2)})^V.$$

Main idea

The class $L^{(2)}$ can be changed only by changing the sets of ordinals. HOD, in a choiceless universe, can be changed by adjoining non-well-orderable objects even when no set of ordinals is added.

4.2 The two-model criterion

Proposition 4.3. *Suppose $N_1 \subseteq N_2$ are transitive models of ZF with the same sets of ordinals, and suppose a real $a \in N_1$ satisfies*

$$a \notin \mathrm{HOD}^{N_1} \qquad \textit{and} \qquad a \in \mathrm{HOD}^{N_2}.$$

Then

$$N_2 \models L^{(2)} \subsetneq \mathrm{HOD}.$$

Proof. By Theorem 4.2, the two models have the same $L^{(2)}$. By Theorem 2.2, $a \notin \mathrm{HOD}^{N_1}$ implies $a \notin (L^{(2)})^{N_1}$, hence $a \notin (L^{(2)})^{N_2}$. But $a \in \mathrm{HOD}^{N_2}$. □

5 The symmetric system

5.1 The ground model and the cardinal gaps

For readability, work over a countable transitive model

$$M \models \mathsf{ZFC} + V = L.$$

This is the usual expository presentation. The relative consistency statement from Con(ZF) is obtained by carrying the forcing construction out over a countable model of $\mathsf{ZF} + V = L$; no transitive-model hypothesis is part of the final theorem.

Fix a parameter-free definable sequence

$$\langle \kappa_n : n < \omega \rangle$$

of regular L-cardinals such that

$$\left|\mathcal{P}^{<\omega}(\kappa_n)\right|^L < \kappa_{n+1} \qquad (n < \omega), \tag{4}$$

where

$$\mathcal{P}^0(\mu) = \mu, \qquad \mathcal{P}^{r+1}(\mu) = \mathcal{P}(\mathcal{P}^r(\mu)), \qquad \mathcal{P}^{<\omega}(\mu) = \bigcup_{r<\omega} \mathcal{P}^r(\mu).$$

Szczepaniak's literal choice is

$$\kappa_n = \aleph^L_{\omega+\omega n+76}.$$

The number 76 has no mathematical significance. Under GCH, the gap of ω many successor cardinals between consecutive terms is what guarantees Equation (4).

5.2 A matrix of generic subsets at each cardinal

For regular κ, define

$$\mathbb{Q}(\kappa) = \mathrm{Fn}_{<\kappa}(\kappa \times \kappa, 2).$$

Thus a condition is a partial function

$$p : \kappa \times \kappa \longrightarrow 2$$

of size $< \kappa$, ordered by reverse inclusion. Put

$$\mathbb{P} = \prod_{n<\omega} \mathbb{Q}(\kappa_n),$$

where the product is the full-support product computed in L.

If $G \subseteq \mathbb{P}$ is generic, its nth coordinate gives a matrix

$$g_n : \kappa_n \times \kappa_n \longrightarrow 2.$$

For $\xi < \kappa_n$, define the ξth row

$$c^n_\xi = \{\eta < \kappa_n : g_n(\xi, \eta) = 1\}.$$

Thus the nth factor adds the indexed family

$$\langle c^n_\xi : \xi < \kappa_n \rangle$$

of subsets of κ_n.

The basic closure facts are:

- $\mathbb{Q}(\kappa_n)$ is $< \kappa_n$-closed;
- $\mathbb{P}$ is countably closed, hence adds no reals;
- the tail $\mathbb{P}_{>n} = \prod_{k>n} \mathbb{Q}(\kappa_k)$ is $< \kappa_{n+1}$-closed;
- the finite initial product $\mathbb{P}_{\leq n} = \prod_{k \leq n} \mathbb{Q}(\kappa_k)$ has size at most κ_n.

5.3 Permuting rows

Let

$$\Gamma = \prod_{n<\omega} \mathrm{Sym}(\kappa_n).$$

For $\pi = \langle \pi_n : n < \omega \rangle \in \Gamma$, define πp by permuting only the first coordinate:

$$(\pi p)_n(\pi_n(\xi), \eta) = p_n(\xi, \eta).$$

Let

$$E = \bigcup_{n<\omega} (\{n\} \times \kappa_n).$$

For finite $e \subseteq E$, put

$$\mathrm{Fix}(e) = \{\pi \in \Gamma : \pi_n(\xi) = \xi \text{ whenever } (n, \xi) \in e\}.$$

Let $\mathcal{F}$ be the normal filter of subgroups generated by the groups $\mathrm{Fix}(e)$. The associated symmetric model is denoted

$$M[G/\mathcal{F}],$$

the usual notation for the interpretations of the hereditarily $\mathcal{F}$-symmetric names.

For a condition p and finite $e \subseteq E$, let $p \restriction e$ be the part of p using only the rows listed in e.

Lemma 5.1 (Restriction lemma). *Let $\dot{x}$ be a name and suppose*

$$\mathrm{Fix}(e) \subseteq \mathrm{sym}(\dot{x})$$

for finite $e \subseteq E$. Then, for every formula φ,

$$p \Vdash \varphi(\dot{x}) \quad \Longleftrightarrow \quad p \restriction e \Vdash \varphi(\dot{x}).$$

The same statement holds with any finite tuple of names supported by e.

Proof. The right-to-left implication follows from $p \leq p \restriction e$. For the other direction, suppose $q \leq p \restriction e$. At coordinate n, fewer than κ_n many rows occur in p_n or q_n. Choose a permutation π_n which fixes every row protected by e and moves all other rows used by p_n away from those used by q_n. Then q is compatible with πp, and $\pi \in \mathrm{Fix}(e)$. Since π fixes $\dot{x}$, the symmetry lemma gives

$$p \Vdash \varphi(\dot{x}) \quad \Longleftrightarrow \quad \pi p \Vdash \varphi(\dot{x}).$$

Every extension of $p \restriction e$ is therefore compatible with a condition forcing $\varphi(\dot{x})$, so $p \restriction e$ forces it. □

5.4 The unordered generic blocks

For each n, put

$$S_n = \{c^n_\xi : \xi < \kappa_n\},$$

and let

$$\mathcal{Y} = \{c^n_\xi : n < \omega,\ \xi < \kappa_n\}.$$

Every individual row c^n_ξ has the one-point support $\{(n, \xi)\}$, while S_n is fixed as a set by every row permutation. Hence all of these objects belong to the symmetric model.

Lemma 5.2 (Finite-row capture). *Let x be a set of ordinals in $M[G/\mathcal{F}]$. Then there is a finite set $t \subseteq \mathcal{Y}$ such that*

$$x \in M[t].$$

The same conclusion holds over any enlarged ground model on which the row permutations act trivially.

Proof. Choose a hereditarily symmetric name $\dot{x}$ with finite support e. Replace it by the canonical name

$$\dot{x}' = \{\langle \check{\alpha}, p \rangle : p \Vdash \check{\alpha} \in \dot{x}\}.$$

By Theorem 5.1, membership in $\dot{x}'$ depends only on $p \restriction e$. The restricted generic filter is recoverable from the finitely many rows

$$t = \{c^n_\xi : (n, \xi) \in e\}.$$

Thus $x \in M[t]$. □

Lemma 5.3 (The blocks are not well-orderable). *In the symmetric model there is no injection*

$$f : S_n \longrightarrow \mathrm{Ord}.$$

Proof. Suppose a condition p forces that $\dot{f} : S_n \to \mathrm{Ord}$ is injective, and let e support $\dot{f}$. Choose $\xi < \kappa_n$ outside e and strengthen p to a condition q deciding

$$\dot{f}(c^n_\xi) = \alpha.$$

Choose $\eta \neq \xi$ outside e and outside every row used by q. Let π swap ξ and η at coordinate n and fix everything else. Then q and πq are compatible, while

$$\pi q \Vdash \dot{f}(c^n_\eta) = \alpha.$$

A common extension contradicts injectivity. □

Remark 5.4 (Comparison with the first Cohen model). At every κ_n we see the familiar first-Cohen-model phenomenon. The rows c^n_ξ exist individually, and their set S_n exists, but the symmetric model cannot attach ordinal labels to all members of S_n at once.

6 The two intermediate models

6.1 Adding an independent Cohen real

Let

$$\mathbb{C} = \mathrm{Add}(\omega, 1),$$

and let $A \times G$ be $\mathbb{C} \times \mathbb{P}$-generic over M. Write

$$a = \bigcup A \subseteq \omega$$

for the Cohen real. Since the symmetry group acts only on the $\mathbb{P}$-coordinate, we may form

$$\mathsf{W}_0 = M[G/\mathcal{F}] \qquad \text{and} \qquad \mathsf{W} = M[a][G/\mathcal{F}].$$

For each n, let

$$Z_n = \mathcal{P}^{\mathsf{W}_0}(\kappa_n) = \mathcal{P}(\kappa_n) \cap \mathsf{W}_0, \qquad Z = \langle Z_n : n < \omega \rangle.$$

Thus Z_n is the entire power set of κ_n as computed in the symmetric model without the Cohen real. In particular every row c^n_ξ belongs to Z_n.

Use a to select some of the unordered blocks:

$$B = \{\langle n, S_n \rangle : n \in a\}.$$

The labels n are part of the object B. Finally define

$$\mathsf{U} = M[Z], \qquad N_1 = M[a, Z] = \mathsf{U}[a], \qquad N_2 = M[a, Z, B].$$

Here $M[X_0, \ldots, X_k]$ denotes the standard constructible closure of M and the displayed parameters inside the ambient symmetric extension. These are transitive inner models of ZF.

6.2 The models have the same sets of ordinals

Lemma 6.1. *The models N_1, N_2, and W have exactly the same sets of ordinals. In particular,*

$$\mathcal{P}(\mathrm{On})^{N_1} = \mathcal{P}(\mathrm{On})^{N_2}.$$

Proof. The inclusions

$$N_1 \subseteq N_2 \subseteq \mathsf{W}$$

are clear. Let x be a set of ordinals in W. Apply Theorem 5.2 over the enlarged ground $M[a]$. There is a finite set of rows $t \subseteq \mathcal{Y}$ such that

$$x \in M[a, t].$$

Every member of t belongs to some Z_n, hence belongs to N_1. Therefore $M[a, t] \subseteq N_1$, and $x \in N_1$. □

By Theorem 4.2,

$$(L^{(2)})^{N_1} = (L^{(2)})^{N_2}. \tag{5}$$

6.3 The Cohen real is not HOD in N_1

Lemma 6.2.

$$a \notin \mathrm{HOD}^{N_1}.$$

Proof. The Cohen filter A is generic over $M[G]$, hence over the inner model $\mathsf{U} = M[Z] \subseteq M[G]$. Thus

$$N_1 = \mathsf{U}[a]$$

is a homogeneous Cohen extension of U. The standard homogeneity argument shows that every ordinal-definable set of ordinals in $\mathsf{U}[a]$ belongs to U. Since $a \notin \mathsf{U}$, the real a is not ordinal definable in N_1 and therefore is not in HOD^{N_1}. □

The remaining task is to show that B makes a definable in N_2 without adding any set of ordinals.

7 Localizing low-rank information

The nth bit of a will be read from the passage between the power-set structures at κ_n and κ_{n+1}. To make this work, one must know that low-rank objects cannot depend on the remote tail of the product.

7.1 Small forcing followed by a closed tail

The following is a modern version of Szczepaniak's Fact 17. It combines Easton's lemma with Grigorieff's intersection theorem for intermediate models.

Lemma 7.1 (Small forcing–closed tail lemma)**.** *Suppose*

$$M \models \mathsf{ZFC} + \mathsf{GCH},$$

$\mathbb{R}_0, \mathbb{Q}_0 \in M$*, and* $E \times H$ *is* $\mathbb{R}_0 \times \mathbb{Q}_0$*-generic over* M*. Let* $\mu < \kappa$ *be cardinals such that*

1. $|\mathbb{R}_0| \leq \mu$*;*
2. $\mathbb{Q}_0$ *is* $< \kappa$*-closed in* M*;*
3. *in* $M[E]$*, the set* $\mathcal{P}^{<\omega}(\mu)$ *has size* $< \kappa$*.*

Let

$$M \subseteq N \subseteq M[E]$$

be an intermediate model, and let

$$N \subseteq K \subseteq N[H]$$

be an inner model of the full product extension. Then

$$\mathcal{P}^{<\omega}(\mu) \cap K \in N.$$

Proof. Because $|\mathbb{R}_0| < \kappa$, the forcing $\mathbb{R}_0$ is κ-c.c. Easton's lemma implies that, in $M[E]$, the forcing $\mathbb{Q}_0$ is $< \kappa$-distributive. Hence it adds no subsets of any set of size $< \kappa$. Therefore

$$D = \mathcal{P}^{<\omega}(\mu) \cap K$$

belongs to $M[E]$. On the other hand, $D \in K \subseteq N[H]$.

Grigorieff's product-intersection theorem gives

$$M[E] \cap N[H] = N.$$

Indeed, if $N \subseteq N[D] \subseteq M[E]$, then $N[D][H] = N[H]$, and the intermediate-model theorem forces $N[D] = N$. Consequently $D \in N$. □

Remark 7.2 (Why the cardinal gaps matter)*.* The closure is used only after the small initial forcing has been performed. Easton's lemma converts ground-model closure into distributivity in the small-forcing extension. This is the reason the regular cardinals κ_n are chosen so far apart.

7.2 Application to the product

Write

$$\mathbb{P}_{\leq n} = \prod_{k \leq n} \mathbb{Q}(\kappa_k), \qquad \mathbb{P}_{>n} = \prod_{k > n} \mathbb{Q}(\kappa_k).$$

The tail $\mathbb{P}_{>n}$ is $< \kappa_{n+1}$-closed, while

$$|\mathbb{C} \times \mathbb{P}_{\leq n}| \leq \kappa_n.$$

The gap condition Equation (4) therefore lets us apply Theorem 7.1.

A standard closure argument also shows that

$$Z_n \in M[G \restriction (n+1)].$$

Intuitively, the tail beyond n is too closed to alter the complete collection of symmetric subsets of κ_n. Likewise, every selected block S_k with $k \leq n$ belongs to the initial extension.

For $r < \omega$, all the earlier Z_k with $k < n$ are recoverable from Z_n by

$$Z_k = \{x \in Z_n : x \subseteq \kappa_k\}.$$

Thus the model generated by Z_n already contains the earlier part of Z.

Lemma 7.3 (Rank localization). *For every $n < \omega$, let*

$$D_n = \mathcal{P}^{<\omega}(\kappa_n) \cap N_2.$$

Then

$$D_n \in M[a, Z_n, B \restriction (n+1)].$$

Here

$$B \restriction (n+1) = \{\langle k, S_k \rangle : k \leq n \text{ and } k \in a\}.$$

Proof. Apply Theorem 7.1 with

$$\mathbb{R}_0 = \mathbb{C} \times \mathbb{P}_{\leq n}, \qquad \mathbb{Q}_0 = \mathbb{P}_{>n},$$

$$N = M[a, Z_n, B \restriction (n+1)], \qquad K = N_2.$$

The model N lies in the initial extension, and N_2 lies between N and $N[G \restriction (\omega \setminus (n+1))]$. The conclusion is exactly the displayed statement. □

8 Recovering the sequence Z from the sets of ordinals

To prove the easy half of the bit-decoding criterion, we need

$$Z \in L(\mathcal{P}(\mathrm{On})),$$

where $L(\mathcal{P}(\mathrm{On}))$ is computed in N_2 throughout this section. This is the most compressed step in Szczepaniak's paper. We first isolate two standard facts about Cohen forcing.

8.1 Two Cohen lemmas

The original paper does not isolate the next statement. It is the standard Cohen intermediate-model fact tacitly used in the one-line verification of the formula defining Z_m. We record it separately because this is otherwise the least transparent part of the proof.

Lemma 8.1 (A new Cohen-intermediate set yields a new constructible real). *Let U be an inner model of a countably closed forcing extension of L, and let c be Cohen-generic over U. If $x \in U[c] \setminus U$ is a set of ordinals, then*

$$\mathbb{R} \cap L[x] \not\subseteq L.$$

In other words, $L[x]$ contains a nonconstructible real.

Proof sketch. Let $\mathbb{B}$ be the Cohen algebra and let $\mathbb{B}_x$ be the complete subalgebra corresponding, by the intermediate-model theorem, to the extension $U \subseteq U[x] \subseteq U[c]$. Since $x \notin U$, this subalgebra is nontrivial. Every complete subalgebra of the Cohen algebra is separable, so the induced generic filter can be coded by a real r.

The countable dense subalgebra and the map recovering its generic from x are coded by a countable sequence of ordinals and Cohen-algebra elements belonging to U. Because U lies in a countably closed extension of L, it has no new countable sequences of ordinals over L. The code is therefore already in L, and the generic trace satisfies

$$r \in L[x] \setminus L.$$

This is the usual Boolean-algebra proof of the intermediate-model theorem, specialized to Cohen forcing. □

Lemma 8.2 (Solovay)**.** *If y is Cohen-generic over $L[x]$, then*

$$\mathbb{R} \cap L[x] \cap L[y] = \mathbb{R} \cap L.$$

Proof. Only the nontrivial inclusion requires proof. Let $r \in \mathbb{R} \cap L[x] \cap L[y]$. Since y is Cohen-generic over L, choose a Cohen name $\tau \in L$ with $r = \tau^y$. By the truth lemma over $L[x]$, some condition $p \subseteq y$ forces

$$\tau = \check{r}.$$

For each $k < \omega$, the Boolean value of "$k \in \tau$" is computed from the L-name τ inside L. Hence

$$r = \{k < \omega : p \Vdash k \in \tau\}$$

belongs to L. □

8.2 An internal definition of Z_m

Let $\varphi_m(x)$ be the following assertion, evaluated in N_2:

> $x \subseteq \kappa_m$, and there is a real y which is Cohen-generic over $L[x]$ such that every real of the ambient universe belongs to $L[y]$.

In symbols,

$$\varphi_m(x) \iff x \subseteq \kappa_m \ \wedge\ \exists y \in 2^\omega \left(y \text{ is Cohen-generic over } L[x] \ \wedge\ \mathbb{R}^{N_2} \subseteq L[y]\right). \tag{6}$$

Proposition 8.3. *For every $m < \omega$,*

$$Z_m = \{x \subseteq \kappa_m : N_2 \models \varphi_m(x)\}.$$

Consequently,

$$Z \in L(\mathcal{P}(\mathrm{On})).$$

Proof. First suppose $x \in Z_m$. Then $x \in \mathsf{W}_0 \subseteq M[G]$, so $L[x] \subseteq M[G]$. The real a is Cohen-generic over $M[G]$, hence over $L[x]$. Since $\mathbb{P}$ is countably closed, it adds no reals, and therefore

$$\mathbb{R}^{N_2} \subseteq \mathbb{R}^{M[a]} = \mathbb{R}^{L[a]}.$$

Thus $y = a$ witnesses $\varphi_m(x)$.

Conversely, suppose $x \subseteq \kappa_m$ belongs to N_2 and satisfies φ_m, witnessed by y. By Theorem 6.1, every set of ordinals in N_2 already belongs to

$$N_1 = \mathsf{U}[a], \qquad \mathsf{U} = M[Z].$$

If $x \notin \mathsf{U}$, then Theorem 8.1 gives a real

$$r \in L[x] \setminus L.$$

The last clause of φ_m puts r in $L[y]$. But y is Cohen-generic over $L[x]$, so Theorem 8.2 implies $r \in L$, a contradiction. Hence $x \in \mathsf{U}$. Since $\mathsf{U} \subseteq \mathsf{W}_0$ and $x \subseteq \kappa_m$, the definition of Z_m gives $x \in Z_m$.

It remains to place Z in $L(\mathcal{P}(\mathrm{On}))$. The models N_2 and $L(\mathcal{P}(\mathrm{On}))$ have the same sets of ordinals. In particular they have the same reals and the same subsets of κ_m. The statement that y is Cohen-generic over $L[x]$ is absolute between them, and both compute the same $L[x]$ and the same collection of ambient reals. Hence the definition Equation (6) is absolute. Each Z_m is therefore definable inside $L(\mathcal{P}(\mathrm{On}))$, uniformly in m, and Replacement gives the sequence Z. $\square$

9 Reading the Cohen real from successive power sets

For each $n < \omega$, let

$$Y_n = \mathcal{P}^{2,N_2}(\kappa_n) = \mathcal{P}^{N_2}(\mathcal{P}^{N_2}(\kappa_n)).$$

The central coding lemma is the following.

Theorem 9.1 (Bit-decoding criterion)**.** *For every $n < \omega$,*

$$n + 1 \in a \iff Y_{n+1} \notin L(\mathcal{P}(\mathrm{On}))[Y_n]. \tag{7}$$

9.1 The easy direction

Assume $n + 1 \notin a$. Then no new block is selected at stage $n + 1$, so

$$B \restriction (n + 2) = B \restriction (n + 1).$$

By Theorem 7.3,

$$Y_{n+1} \in M[a, Z_{n+1}, B \restriction (n + 1)].$$

Now:

- $Z_{n+1} \in L(\mathcal{P}(\mathrm{On}))$ by Theorem 8.3;
- $a \in Y_n$: every natural number is itself a subset of κ_n, so $a \subseteq \mathcal{P}(\kappa_n)$ and hence $a \in \mathcal{P}^2(\kappa_n) = Y_n$;

- the finite sequence $B \restriction (n+1)$ has a uniform code in Y_n.

For the last point, use a fixed injection

$$(n+1) \times \mathcal{P}(\kappa_n) \longrightarrow \mathcal{P}(\kappa_n)$$

to code the finitely many sets S_k as one subset of $\mathcal{P}(\kappa_n)$. Therefore

$$Y_{n+1} \in L(\mathcal{P}(\mathrm{On}))[Y_n].$$

This proves

$$n+1 \notin a \quad \Longrightarrow \quad Y_{n+1} \in L(\mathcal{P}(\mathrm{On}))[Y_n].$$

9.2 The diagonal twist

Now assume $n+1 \in a$. Define an automorphism Δ of $\mathbb{P}$ which is the identity at every coordinate except $n+1$ and, at that coordinate, flips the diagonal bits:

$$(\Delta p)_{n+1}(\xi,\eta) = \begin{cases} 1 - p_{n+1}(\xi,\xi), & \eta = \xi \text{ and } (\xi,\xi) \in \mathrm{dom}(p_{n+1}), \\ p_{n+1}(\xi,\eta), & \eta \neq \xi. \end{cases}$$

Let

$$H = \Delta``G.$$

Then H is $\mathbb{P}$-generic over $M[a]$. Write

$$d^k_\xi$$

for the rows generated by H. For $k \neq n+1$,

$$d^k_\xi = c^k_\xi,$$

while

$$d^{n+1}_\xi = c^{n+1}_\xi \triangle \{\xi\}. \tag{8}$$

Let

$$\mathsf{W}^\Delta = M[a][H/\mathcal{F}]$$

be the symmetric model built from H using the same row-permutation system.

Lemma 9.2. *The models* W *and* W^Δ *have the same sets of ordinals. Hence they have the same class* $L(\mathcal{P}(\mathrm{On}))$.

Proof. By finite-row capture, every set of ordinals in either model belongs to an extension generated by finitely many rows. Replacing a G-row by the corresponding H-row changes it, by Equation (8), by at most the ground-model singleton $\{\xi\}$. Thus corresponding finite tuples of rows generate exactly the same extension of $M[a]$. The collections of sets of ordinals in the two symmetric models are therefore identical. $\square$

Let

$$S^G_{n+1} = \{c^{n+1}_\xi : \xi < \kappa_{n+1}\}, \qquad S^H_{n+1} = \{d^{n+1}_\xi : \xi < \kappa_{n+1}\}.$$

The second of these is the canonical unordered block in W^Δ.

Lemma 9.3.

$$S^G_{n+1} \notin \mathsf{W}^\Delta.$$

Proof. Suppose $S^G_{n+1} \in \mathsf{W}^\Delta$. For $y \in S^H_{n+1}$, define

$$f(y) = \text{the unique } \xi < \kappa_{n+1} \text{ for which some } x \in S^G_{n+1} \text{ satisfies } x \triangle y = \{\xi\}.$$

Existence follows from Equation (8): for $y = d^{n+1}_\xi$, take $x = c^{n+1}_\xi$. For uniqueness, fix distinct $\xi, \zeta < \kappa_{n+1}$ and $m < \omega$. Below any condition, choose m fresh columns on the two rows and extend the condition so that the two rows have opposite values at each of those columns. Thus the set of conditions forcing

$$|c^{n+1}_\xi \triangle c^{n+1}_\zeta| \geq m$$

is dense. In particular, two distinct G-rows differ at more than two coordinates. Toggling the single diagonal point in $d^{n+1}_\xi = c^{n+1}_\xi \triangle \{\xi\}$ therefore cannot make a distinct row c^{n+1}_ζ differ from d^{n+1}_ξ by a singleton. Hence the only possible witness is $x = c^{n+1}_\xi$, and the singleton is $\{\xi\}$. Consequently $f(d^{n+1}_\xi) = \xi$, so f is an injection

$$S^H_{n+1} \longrightarrow \kappa_{n+1}.$$

This contradicts Theorem 5.3 inside W^Δ. $\square$

9.3 The hard direction

By Theorem 7.3, the parameter Y_n belongs to

$$M[a, Z_n, B \restriction (n+1)].$$

The twist occurs only at coordinate $n+1$, so all the displayed parameters are also in W^Δ. By Theorem 9.2, W^Δ has the same $L(\mathcal{P}(\mathrm{On}))$ as N_2. Therefore the relative constructible class

$$L(\mathcal{P}(\mathrm{On}))[Y_n]$$

can be constructed inside W^Δ as well.

Suppose toward a contradiction that

$$Y_{n+1} \in L(\mathcal{P}(\mathrm{On}))[Y_n].$$

Then $Y_{n+1} \in \mathsf{W}^\Delta$. Since $n+1 \in a$, the object B contains S^G_{n+1}, and

$$S^G_{n+1} \subseteq \mathcal{P}^{N_2}(\kappa_{n+1}).$$

Thus

$$S^G_{n+1} \in Y_{n+1}.$$

Transitivity of W^Δ would imply $S^G_{n+1} \in \mathsf{W}^\Delta$, contradicting Theorem 9.3. Hence

$$Y_{n+1} \notin L(\mathcal{P}(\mathrm{On}))[Y_n].$$

Together with the easy direction, this proves Theorem 9.1.

9.4 The real a is HOD in N_2

The sequence $\langle \kappa_n : n < \omega \rangle$ is parameter-free definable from L, the class $L(\mathcal{P}(\mathrm{On}))$ is definable from the sets of ordinals, and $Y_n = \mathcal{P}^2(\kappa_n)$ is definable from n. Hence Equation (7) gives a parameter-free definition of

$$a \cap \{1, 2, 3, \dots\}.$$

The possible zeroth bit is harmless. Let the displayed tail be b. If $0 \notin a$, then $a = b$; if $0 \in a$, then $a = b \cup \{0\}$. Both sets on the right are ordinal definable, so in either case a is ordinal definable. Therefore

$$a \in \mathrm{HOD}^{N_2}.$$

10 Completion of the consistency proof

We now have:

$$\begin{aligned} \mathcal{P}(\mathrm{On})^{N_1} &= \mathcal{P}(\mathrm{On})^{N_2}, \\ a &\notin \mathrm{HOD}^{N_1}, \\ a &\in \mathrm{HOD}^{N_2}. \end{aligned}$$

By Theorem 4.2,

$$(L^{(2)})^{N_1} = (L^{(2)})^{N_2}.$$

By Theorem 2.2,

$$a \notin \mathrm{HOD}^{N_1} \quad \Longrightarrow \quad a \notin (L^{(2)})^{N_1},$$

and therefore $a \notin (L^{(2)})^{N_2}$. Thus

$$a \in \mathrm{HOD}^{N_2} \setminus (L^{(2)})^{N_2}.$$

We have proved:

Theorem 10.1. *If ZF is consistent, then so is*

$$\mathsf{ZF} + L^{(2)} \subsetneq \mathrm{HOD}.$$

11 What the construction teaches us

11.1 Why $L^{(2)}$ is insensitive to B

The object B is not a set of ordinals and is not well-orderable in any obvious uniform way. Nevertheless it carries the selection pattern a. Adjoining B changes no set of ordinals, so by Theorem 4.1 it cannot change $L^{(2)}$.

11.2 Why HOD is sensitive to B

The selected block S_{n+1} determines whether Y_{n+1} belongs to $L(\mathcal{P}(\mathrm{On}))[Y_n]$. The diagonal twist proves that this change is genuine: if the selected block could be reconstructed without being present, it would well-order the canonical unordered block in a twisted symmetric model. Thus the pattern of selected blocks becomes an ordinal definition of a.

11.3 How this defeats the Myhill–Scott proof

The ZFC proof in Section 3 works by placing a copy of a suitable V_α inside a well-orderable stage of $L^{(2)}$. In the choiceless model, the unordered families S_n carry information that is not uniformly ordinal-codeable. They can therefore change what is ordinal definable without changing the sets of ordinals, and hence without changing $L^{(2)}$.

The guiding slogan is

$$\boxed{L^{(2)} \text{ is controlled by sets of ordinals, whereas HOD need not be.}}$$

Choice closes this gap by making the relevant set-sized information well-orderable and hence available for the copy-of-V_α argument.

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