SEPARATORS OF COMPLETE COANALYTIC PAIRS

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Abstract. The present note is merely an exercise in translating the proof of the following result (by Goh) into more classical descriptive set theoretic terms: if (A_0, A_1) is a complete pair of disjoint coanalytic subsets of a Polish space, then every coanalytic set separating A_0 from A_1 is $\mathbf{\Pi}^1_1$-complete. Neither the proof nor the result is novel.

1. Introduction

Let X be a Polish space. A pair (A_0, A_1) of disjoint coanalytic subsets of X is *complete* if, whenever B_0, B_1 are disjoint coanalytic subsets of a zero-dimensional Polish space Y, there is a continuous map $f : Y \to X$ such that

$$f^{-1}(A_i) = B_i \qquad (i = 0, 1).$$

A set $C \subseteq X$ separates A_0 from A_1 if

$$A_0 \subseteq C \subseteq X \setminus A_1.$$

A coanalytic set $C \subseteq X$ is $\mathbf{\Pi}^1_1$-complete if every coanalytic subset of the Baire space is continuously reducible to C.

Saint Raymond proved that the pair consisting of the well-founded trees and the trees with exactly one infinite branch is a complete pair of coanalytic sets [3]. Goh subsequently proved, by recursion-theoretic methods, that every coanalytic separator of this pair is $\mathbf{\Pi}^1_1$-complete [1]. The present note is an exercise in translating Goh's argument into more classical DST terms. Neither the proof nor the result is new.

Completeness of the pair alone does not immediately settle the separator question. Reducing $(B, \varnothing)$ to (A_0, A_1) proves that A_0 is $\mathbf{\Pi}^1_1$-complete, but if C is a larger separator, the same reduction places no restriction on whether points outside B are mapped into C. The argument below supplies the missing control by a diagonal construction.

2. Universal sets and a fixed-point lemma

Write $\mathcal{N} = \omega^\omega$. If $P \subseteq \mathcal{N} \times X$ and $\alpha \in \mathcal{N}$, write

$$P_\alpha = \{x \in X : (\alpha, x) \in P\}.$$

Fix a homeomorphism

$$\langle \cdot, \cdot \rangle : \mathcal{N}^2 \longrightarrow \mathcal{N},$$

and write $\gamma \mapsto ((\gamma)_0, (\gamma)_1)$ for its inverse.

We use a good universal system for $\mathbf{\Pi}^1_1$. Thus, for each Polish space X occurring below, fix a coanalytic set

$$\mathcal{U}^X \subseteq \mathcal{N} \times X$$

whose sections are exactly the coanalytic subsets of X, together with a continuous map

$$s^X : \mathcal{N} \times \mathcal{N} \longrightarrow \mathcal{N}$$

such that

$$\mathcal{U}^X_{s^X(\alpha,\beta)} = (\mathcal{U}^{\mathcal{N}\times X}_\alpha)_\beta \qquad (\alpha, \beta \in \mathcal{N}). \tag{1}$$

Such systems are standard; they may be obtained from the usual universal closed-set codes. Compare the Good Parametrization Lemma [2, 3H.1].

We also use the reduction property of $\mathbf{\Pi}^1_1$: if P_0, P_1 are coanalytic subsets of a Polish space, then there are disjoint coanalytic sets $V_i \subseteq P_i$ such that

$$V_0 \cup V_1 = P_0 \cup P_1.$$

See [2, 4B.10].

Lemma 2.1. *There are disjoint coanalytic sets $V_0, V_1 \subseteq \mathcal{N} \times \mathcal{N}$ with the following property. For any coanalytic sets $R_0, R_1 \subseteq \mathcal{N} \times \mathcal{N}$, there is a continuous map $\theta : \mathcal{N} \to \mathcal{N}$ such that, for every $z \in \mathcal{N}$ for which $(R_0)_z \cap (R_1)_z = \varnothing$,*

$$(V_i)_{\theta(z)} = (R_i)_z \qquad (i = 0, 1).$$

Proof. Define coanalytic sets $P_0, P_1 \subseteq \mathcal{N} \times \mathcal{N}$ by

$$(\alpha, x) \in P_i \quad \iff \quad x \in \mathcal{U}^{\mathcal{N}}_{(\alpha)_i} \qquad (i = 0, 1).$$

By the reduction property, there are disjoint coanalytic sets $V_i \subseteq P_i$ such that

$$V_0 \cup V_1 = P_0 \cup P_1.$$

Whenever $(P_0)_\alpha$ and $(P_1)_\alpha$ are disjoint, one has

$$(V_i)_\alpha = (P_i)_\alpha \qquad (i = 0, 1). \tag{2}$$

Indeed, if $x \in (P_i)_\alpha$, then the equality of the unions places x in one of $(V_0)_\alpha, (V_1)_\alpha$. Since $V_j \subseteq P_j$ and the two P-sections are disjoint, x must belong to $(V_i)_\alpha$. The reverse inclusion follows from $V_i \subseteq P_i$.

Now let $R_0, R_1 \subseteq \mathcal{N} \times \mathcal{N}$ be coanalytic, and choose $\rho_i \in \mathcal{N}$ such that

$$R_i = \mathcal{U}^{\mathcal{N}\times\mathcal{N}}_{\rho_i}.$$

By (1), the maps

$$r_i(z) = s^{\mathcal{N}}(\rho_i, z)$$

are continuous and satisfy

$$\mathcal{U}^{\mathcal{N}}_{r_i(z)} = (R_i)_z.$$

Put

$$\theta(z) = \langle r_0(z), r_1(z) \rangle.$$

Then $(P_i)_{\theta(z)} = (R_i)_z$. If the latter two sections are disjoint, the conclusion follows from (2). $\square$

The following is the continuous fixed-point lemma associated with a good universal system. Equality in its conclusion is equality of the sets coded by the two parameters, not necessarily equality of the parameters themselves.

Lemma 2.2. *Let X be a Polish space, and let $\sigma : \mathcal{N}\times\mathcal{N} \to \mathcal{N}$ be continuous. There is a continuous map $a : \mathcal{N} \to \mathcal{N}$ such that*

$$\mathcal{U}^X_{a(y)} = \mathcal{U}^X_{\sigma(a(y),y)} \qquad (y \in \mathcal{N}). \tag{3}$$

Proof. Define $R \subseteq \mathcal{N} \times X$ by

$$(\langle e, y\rangle, x) \in R \iff x \in \mathcal{U}^X_{\sigma(s^X(e,\langle e,y\rangle),y)}.$$

The set R is coanalytic. Choose $r \in \mathcal{N}$ such that

$$R = \mathcal{U}^{\mathcal{N}\times X}_r,$$

and define

$$a(y) = s^X(r, \langle r, y\rangle).$$

Then a is continuous, and for every $y \in \mathcal{N}$,

$$\begin{aligned} \mathcal{U}^X_{a(y)} &= \left(\mathcal{U}^{\mathcal{N}\times X}_r\right)_{\langle r,y\rangle} \\ &= R_{\langle r,y\rangle} \\ &= \mathcal{U}^X_{\sigma(s^X(r,\langle r,y\rangle),y)} \\ &= \mathcal{U}^X_{\sigma(a(y),y)}. \end{aligned}$$

$\square$

3. Separators of complete pairs

Theorem 3.1. *Let X be a Polish space, let (A_0, A_1) be a complete pair of disjoint coanalytic subsets of X, and let $C \subseteq X$ be coanalytic. If*

$$A_0 \subseteq C \subseteq X \setminus A_1,$$

then C is $\mathbf{\Pi}^1_1$-complete.

Proof. Let V_0, V_1 be as in Lemma 2.1, and define

$$E_i = \{t \in \mathcal{N} : (t, t) \in V_i\} \qquad (i = 0, 1).$$

The sets E_0, E_1 are disjoint and coanalytic. By completeness of (A_0, A_1), there is a continuous map $h : \mathcal{N} \to X$ such that

$$h^{-1}(A_i) = E_i \qquad (i = 0, 1). \tag{4}$$

We first construct a continuous map $q : \mathcal{N} \to X$ satisfying

$$\mathcal{U}^X_\alpha \cap C = \varnothing \implies q(\alpha) \notin C \cup \mathcal{U}^X_\alpha. \tag{5}$$

Define $R_0, R_1 \subseteq \mathcal{N} \times \mathcal{N}$ by

$$\begin{aligned}(\alpha, t) \in R_0 \quad &\iff \quad h(t) \in A_1 \text{ or } (\alpha, h(t)) \in \mathcal{U}^X, \\ (\alpha, t) \in R_1 \quad &\iff \quad h(t) \in C.\end{aligned}$$

These sets are coanalytic. By Lemma 2.1, there is a continuous $\theta : \mathcal{N} \to \mathcal{N}$ such that, whenever $(R_0)_\alpha$ and $(R_1)_\alpha$ are disjoint,

$$(V_i)_{\theta(\alpha)} = (R_i)_\alpha \qquad (i = 0, 1). \tag{6}$$

Put

$$q(\alpha) = h(\theta(\alpha)).$$

Suppose that $\mathcal{U}^X_\alpha \cap C = \varnothing$. Since $A_1 \cap C = \varnothing$, the sections $(R_0)_\alpha$ and $(R_1)_\alpha$ are disjoint, so (6) applies. Write $t = \theta(\alpha)$. If $q(\alpha) \in C$, then

$$t \in (R_1)_\alpha = (V_1)_t,$$

so $t \in E_1$. By (4), $q(\alpha) = h(t) \in A_1$, contradicting $C \cap A_1 = \varnothing$. Thus $q(\alpha) \notin C$.

If $q(\alpha) \in \mathcal{U}^X_\alpha$, then

$$t \in (R_0)_\alpha = (V_0)_t,$$

so $t \in E_0$. It follows from (4) that $q(\alpha) = h(t) \in A_0 \subseteq C$, contradicting $\mathcal{U}^X_\alpha \cap C = \varnothing$. This proves (5).

Now let $B \subseteq \mathcal{N}$ be an arbitrary coanalytic set, and define $H \subseteq \mathcal{N} \times X$ by

$$(\langle \alpha, y \rangle, x) \in H \quad \iff \quad y \in B \text{ and } x = q(\alpha).$$

The set H is coanalytic: after decoding the first coordinate, it is the intersection of the continuous preimage of B with the graph of the continuous map $\beta \mapsto q((\beta)_0)$, and the latter graph is closed. Choose $\rho \in \mathcal{N}$ such that

$$H = \mathcal{U}^{\mathcal{N} \times X}_\rho,$$

and define

$$\sigma(\alpha, y) = s^X(\rho, \langle \alpha, y \rangle).$$

Then $\sigma : \mathcal{N} \times \mathcal{N} \to \mathcal{N}$ is continuous and

$$\mathcal{U}^X_{\sigma(\alpha,y)} = \begin{cases} \{q(\alpha)\}, & y \in B, \\ \varnothing, & y \notin B. \end{cases} \tag{7}$$

Apply Lemma 2.2 to obtain a continuous $a : \mathcal{N} \to \mathcal{N}$ such that

$$\mathcal{U}^X_{a(y)} = \mathcal{U}^X_{\sigma(a(y),y)}.$$

Finally, set

$$f(y) = q(a(y)).$$

If $y \notin B$, then (7) gives $\mathcal{U}^X_{a(y)} = \varnothing$, and (5) gives $f(y) \notin C$. If $y \in B$, then

$$\mathcal{U}^X_{a(y)} = \{q(a(y))\} = \{f(y)\}.$$

If $f(y) \notin C$, this section is disjoint from C, so (5) implies

$$f(y) = q(a(y)) \notin \mathcal{U}^X_{a(y)},$$

a contradiction. Therefore $f(y) \in C$. We have shown that

$$y \in B \iff f(y) \in C.$$

Since f is continuous and B was an arbitrary coanalytic subset of $\mathcal{N}$, the set C is $\mathbf{\Pi}^1_1$-complete. □

Remark 3.2. The proof uses only the following properties of $\mathbf{\Pi}^1_1$: it contains the closed sets, is closed under finite unions, finite intersections, and continuous preimages, has the reduction property, and admits a good universal system with continuous sectioning maps as in (1). Consequently, the same argument applies to any boldface pointclass with these properties: every separator in the pointclass of a complete disjoint pair in the pointclass is complete for the pointclass.

Corollary 3.3. *Let* Tr *be the Polish space of trees on* ω*, and let*

$$\mathrm{WF} = \{T \in \mathrm{Tr} : [T] = \varnothing\}, \qquad \mathrm{UB} = \{T \in \mathrm{Tr} : |[T]| = 1\}.$$

If $C \subseteq \mathrm{Tr}$ *is coanalytic and*

$$\mathrm{WF} \subseteq C \subseteq \mathrm{Tr} \setminus \mathrm{UB},$$

then C *is* $\mathbf{\Pi}^1_1$*-complete.*

Proof. Saint Raymond proved that $(\mathrm{WF}, \mathrm{UB})$ is a complete pair of coanalytic sets [3, Theorem 23]. Apply Theorem 3.1. □

Remark 3.4. Saint Raymond denotes UB by WF^*.

References

[1] J. L. Goh, *Inseparable pairs and recursion theory*, Slides for Boise Extravaganza in Set Theory, June 2021; `https://bpb-us-w2.wpmucdn.com/blog.nus.edu.sg/dist/1/16136/files/2022/09/goh_inseparable_slides_BEST.pdf`

[2] Y. N. Moschovakis, *Descriptive Set Theory*, second ed., Mathematical Surveys and Monographs, vol. 155, American Mathematical Society, Providence, RI, 2009.

[3] J. Saint Raymond, *Complete pairs of coanalytic sets*, Fund. Math. **194** (2007), no. 3, 267–281; doi:10.4064/fm194-3-4.