Two questions on E_0-like generic equivalence

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Abstract

We answer Problems 8.1 and 8.3 from [1]. We first note that the condition attributed there to ordinary Prikry forcing is not correct as written: the appropriate formulation uses finite symmetric difference of the ranges of the generic sequences, rather than eventual equality at the same coordinates. We record answers to the two problems for both formulations. A length-ω Magidor forcing gives a genuinely Prikry-type example satisfying both conditions. Rigid real-adding forcing gives a broad source of further examples, and a forcing of Jech and Shelah shows that the condition does imply even an inaccessible cardinal. An E_0-invariant Jensen-type forcing of Kanovei and Lyubetsky gives a stronger nontrivial example in which the generic reals in a fixed extension form exactly one full E_0-class. Finally, a simple recoding turns the real-forcing previous examples into answers for the version of the problems with corrected condition.

1 The two formulations

Following [6], consider a countable model M of a sufficiently large fragment of ZFC and a forcing notion $\mathbb{P} \in M$. We are interested in the equivalence relation $G \sim H \Leftrightarrow M[G] = M[H]$. Section 8 of [1] takes the analysis of Prikry forcing ([2]) as its motivating example, where the generic filter is identified with an increasing generic sequence

$$a = \langle a(n) : n < \omega \rangle.$$

The condition printed there is

$$M[a] = M[b] \quad \Longleftrightarrow \quad \exists N < \omega \ \forall n > N \ (a(n) = b(n)). \tag{$*_{\mathrm{coord}}$}$$

This condition is not correct for ordinary Prikry forcing.[1]

The corresponding statement with ranges is

$$M[a] = M[b] \quad \Longleftrightarrow \quad \big|\mathrm{ran}(a) \triangle \mathrm{ran}(b)\big| < \omega. \tag{$*_{\mathrm{range}}$}$$

This is the most plausible reading of Calderoni–Sinapova's phrasing, phrase "coincide on a tail," [2, Fact 4.2(1)]: the proof of the nontrivial direction assumes that the symmetric difference is infinite and derives a contradiction. The converse is immediate for increasing sequences over

[1] If $a = \langle \alpha_0, \alpha_1, \ldots \rangle$ is Prikry generic over a transitive M, then its shift $b = \langle \alpha_1, \alpha_2, \ldots \rangle$ is again Prikry generic. Moreover $M[a] = M[b]$: one inclusion is immediate, and the other follows because $\alpha_0 \in M$. Since a is strictly increasing, however, $a(n) \neq b(n)$ for every n.

a transitive ground model: if their ranges differ by finitely many ordinals, each sequence is recoverable from the other.

In [1], Problem 8.1 asks for more examples of forcing that satisfy the condition; Problem 8.3 asks if the condition has large cardinal strength. For both the $(*_{\mathrm{coord}})$ and $(*_{\mathrm{range}})$ formulations of the problems, we supply further examples for 8.1 and answer 8.3 negatively.

2 A Magidor-forcing example

The closest example to the intended Prikry example is supplied by the uniqueness theorem for Magidor sequences. We use the formulation of Fuchs [3]. Let

$$\mathbb{M} = \mathbb{M}(\vec{U}, \vec{f})$$

be a Magidor forcing of length α, based on a measurable cardinal κ carrying a sequence $\vec{U} = \langle U_\xi : \xi < \alpha \rangle$ of normal measures increasing in the Mitchell order. Its generic object is an increasing function $c : \alpha \to \kappa$.

Theorem 1 (Fuchs)**.** *The following is [3, Theorem 7.5 and Corollary 7.6]. Let c be $\mathbb{M}$-generic over a transitive ground model M, and let $d \in M[c]$ be a function of the type of an $\mathbb{M}$-generic sequence. Then the following are equivalent:*

1. *d is $\mathbb{M}$-generic over M;*
2. *the set*
$$\{\xi < \alpha : c(\xi) \neq d(\xi)\}$$
is finite and contains no limit ordinal;
3. *$M[c] = M[d]$.*

Proposition 2. *Suppose that M has a measurable cardinal κ carrying a Mitchell-increasing sequence $\langle U_n : n < \omega \rangle$ of normal measures, and let $\mathbb{M} \in M$ be the corresponding Magidor forcing of length ω. For any two $\mathbb{M}$-generic sequences c, d over M,*

$$M[c] = M[d] \iff \exists N < \omega \ \forall n > N \ (c(n) = d(n)).$$

Moreover,

$$M[c] = M[d] \iff \big|\mathrm{ran}(c) \triangle \mathrm{ran}(d)\big| < \omega.$$

Thus length-ω Magidor forcing satisfies both $(*_{\mathrm{coord}})$ and $(*_{\mathrm{range}})$.

Proof. Suppose first that $M[c] = M[d]$. Then $d \in M[c]$, so Theorem 1 applies. It follows in particular that $c(n) = d(n)$ for all but finitely many n. This also implies that the ranges of c and d have finite symmetric difference.

Conversely, suppose that c and d differ at only finitely many coordinates. The finite set

$$F = \{(n, d(n)) : c(n) \neq d(n)\}$$

belongs to M, and d is definable from c and F; the same argument with c and d interchanged gives $M[c] = M[d]$. Likewise, suppose that the ranges have finite symmetric difference. The finite sets

$$A = \mathrm{ran}(c) \setminus \mathrm{ran}(d) \quad \text{and} \quad B = \mathrm{ran}(d) \setminus \mathrm{ran}(c)$$

belong to M, and d is the increasing enumeration of $(\mathrm{ran}(c) \setminus A) \cup B$. Again $M[c] = M[d]$. $\square$

This gives a positive answer to Problem 8.1, in both formulations, within the class of genuinely Prikry-type forcings.

3 Rigid forcing

There is also a general source of ZFC-examples. Recall the Vopěnka–Hájek theorem in the form used by Smythe.

Theorem 3 (Vopěnka–Hájek)**.** *This is the form recorded in [6, Theorem 2.16]. Let $\mathbb{B} \in M$ be a complete Boolean algebra, and let G, H be $\mathbb{B}$-generic filters over M. If $M[G] = M[H]$, then there is an involutive automorphism $e \in \mathrm{Aut}^M(\mathbb{B})$ such that $H = e''G$.*

A complete Boolean algebra is *rigid* if it has no nonidentity automorphism.

Proposition 4. *Let M be a countable transitive model, let $\mathbb{B} \in M$ be a complete Boolean algebra which is rigid in M, and suppose that $\dot{x}$ is a $\mathbb{B}$-name such that*

$$M[G] = M[x_G]$$

for every $\mathbb{B}$-generic filter G over M, where $x_G = \dot{x}^G \in \omega^\omega$. Then for any two such generic filters G, H,

$$M[x_G] = M[x_H] \iff x_G = x_H \iff x_G \, E_0 \, x_H.$$

In particular, the generic reals satisfy $(*_{\mathrm{coord}})$.

Proof. If $M[x_G] = M[x_H]$, then $M[G] = M[H]$. By Theorem 3, there is an automorphism $e \in \mathrm{Aut}^M(\mathbb{B})$ with $H = e''G$. Since $\mathbb{B}$ is rigid in M, e is the identity; hence $G = H$ and $x_G = x_H$.

Equality clearly implies eventual equality. Conversely, if $x_G \, E_0 \, x_H$, then the two reals differ on only finitely many coordinates. The finite modification taking one real to the other is coded by an element of M, so x_G and x_H are interdefinable over M. Thus $M[x_G] = M[x_H]$, and the first part of the proof then gives $x_G = x_H$. □

The hypothesis of this proposition is realized in ZFC.

Theorem 5 (Jech–Shelah [4])**.** *There is a perfect-tree forcing $\mathbb{P}$ which adds a generic branch $g \in \omega^\omega$ such that*

$$V[G] = V[g]$$

and whose complete Boolean algebra $\mathbb{B}(\mathbb{P})$ is rigid.

Corollary 6. *For every suitable countable transitive model M, the Jech–Shelah forcing as computed in M satisfies $(*_{\mathrm{coord}})$. Consequently, Problem 8.1 with the printed condition has a positive answer, whereas Problem 8.3 has a negative answer: the condition does not even imply that M has an inaccessible cardinal.*

Proof. Apply Proposition 4 to the Boolean completion of the Jech–Shelah forcing. The construction and the proof of rigidity are carried out in ZFC and hence can be performed inside any sufficiently strong M. For the ground model, reflect the required finite fragment together with the sentence "there is no inaccessible cardinal" in an appropriate constructible model, and then take a countable elementary submodel and its transitive collapse. By condensation the resulting model is some L_θ with the required properties." □

Remark 7. For a rigid forcing, equality of generic extensions is equality of the corresponding generic filters, and in the situation above it is equality of the generic reals. Thus the example is literal but in this sense rigid: the same-extension classes of generic reals are singletons. The next example shows that this degeneracy is not necessary.

4 A nontrivial E_0-example

Kanovei and Lyubetsky [5] construct in L an E_0-invariant Jensen-type perfect-tree forcing based on Silver forcing. The feature we need is their Lemma 7.4: if G is generic for their forcing over L and x_G is the generic real, then, inside $L[G]$, the reals generic for the same forcing over L are exactly

$$[x_G]_{E_0}.$$

They also have $L[G] = L[x_G]$.

For the formulation of Section 8, let $M = L_\theta$ be a countable transitive model of a fixed finite fragment of ZFC strong enough to carry out the Kanovei–Lyubetsky construction and its proof through Lemma 7.4. Perform their construction internally to M, through ω_1^M, and denote the resulting forcing by $\mathbb{P}^M$.

Lemma 8. *Let $G \subseteq \mathbb{P}^M$ be generic over M, and let x_G be its generic real. Then $M[G] = M[x_G]$, and*

$$\{y \in M[G] \cap 2^\omega : y \text{ is } \mathbb{P}^M\text{-generic over } M\} = [x_G]_{E_0}.$$

Proof. The proofs of Lemmas 7.1 and 7.4 of [5] use the internally constructed sequence of forcing notions and the maximal antichains belonging to the ground model. The same proofs therefore apply with L replaced by M. □

Proposition 9. *If x, y are $\mathbb{P}^M$-generic reals over M, then*

$$M[x] = M[y] \iff x \, E_0 \, y.$$

Consequently $\mathbb{P}^M$ satisfies $(*_{\text{coord}})$.

Proof. If $M[x] = M[y]$, then $y \in M[x]$ and y is $\mathbb{P}^M$-generic over M. By Lemma 8, $y \in [x]_{E_0}$.

Conversely, if $x \, E_0 \, y$, then x and y differ by a finite bit flip coded in M. Hence each belongs to the model generated by the other, so $M[x] = M[y]$. □

Corollary 10. *The printed condition $(*_{\text{coord}})$ can hold nontrivially over a model with no inaccessible cardinal: in a $\mathbb{P}^M$-generic extension, the generic reals over M belonging to that extension form the entire countable E_0-class of the generic real.*

Proof. Choose a countable transitive

$$M = L_\theta \models V = L + \text{"there is no inaccessible cardinal."}$$

which satisfies the finite fragment required for Lemma 8. Such a model is obtained by the same reflection-and-collapse argument used in Corollary 6. Now apply Proposition 9. □

5 Recoding real forcings for the range condition

The real-forcing examples above can be converted uniformly into examples for $(*_{\mathrm{range}})$. For $x \in \omega^\omega$, define

$$c_x(n) = \omega \cdot n + x(n).$$

Then c_x is a strictly increasing sequence of ordinals below ω^2, and x and c_x are uniformly definable from one another. Moreover,

$$x \, E_0 \, y \quad \Longleftrightarrow \quad \left|\operatorname{ran}(c_x) \mathbin{\triangle} \operatorname{ran}(c_y)\right| < \omega. \tag{1}$$

Indeed, $\operatorname{ran}(c_x)$ contains exactly one point in the ordinal interval $[\omega \cdot n, \omega \cdot (n+1))$, namely $\omega \cdot n + x(n)$.

For a finite sequence $s \in \omega^{<\omega}$, let

$$\widehat{s}(i) = \omega \cdot i + s(i).$$

If $T \subseteq \omega^{<\omega}$ is a tree, put

$$\widehat{T} = \{\widehat{s} : s \in T\} \subseteq (\omega^2)^{<\omega}.$$

Thus any tree forcing $\mathbb{P}$ on $\omega^{<\omega}$ has an isomorphic presentation

$$\widehat{\mathbb{P}} = \{\widehat{T} : T \in \mathbb{P}\}$$

whose generic branch corresponding to x is c_x. The transformation is uniform, so $\widehat{\mathbb{P}} \in M$ whenever $\mathbb{P} \in M$.

Proposition 11 (Recoding lemma)**.** *Let $\mathbb{P} \in M$ be a tree forcing whose generic branch $x \in \omega^\omega$ generates the generic extension, and suppose that for any two $\mathbb{P}$-generic branches x, y over M,*

$$M[x] = M[y] \quad \Longleftrightarrow \quad x \, E_0 \, y.$$

Then the isomorphic forcing $\widehat{\mathbb{P}}$ has a strictly increasing generic branch c_x and satisfies

$$M[c_x] = M[c_y] \quad \Longleftrightarrow \quad \left|\operatorname{ran}(c_x) \mathbin{\triangle} \operatorname{ran}(c_y)\right| < \omega$$

for all $\widehat{\mathbb{P}}$-generic branches c_x, c_y over M.

Proof. Since x and c_x are uniformly interdefinable, $M[x] = M[c_x]$. The conclusion now follows from the hypothesis and (1). □

Corollary 12. *Both the Jech–Shelah forcing and the Kanovei–Lyubetsky forcing have isomorphic presentations satisfying $(*_{\mathrm{range}})$. These presentations can be constructed over models of just plain ZFC. Hence Problem 8.1 has a positive answer and Problem 8.3 has a negative answer also for the corrected range condition.*

Proof. For the Jech–Shelah forcing, Proposition 4 gives the hypothesis of Proposition 11; for the Kanovei–Lyubetsky forcing, the hypothesis is Proposition 9. Apply Proposition 11. The ground model may be chosen to satisfy the finite fragments needed for both constructions, as in Corollaries 6 and 10. □

Remark 13. The recoded examples settle the corrected problem under the unrestricted reading in which one asks for a forcing notion with a distinguished increasing generic sequence. They are isomorphic presentations of real forcings. By contrast, Proposition 2 gives a direct example whose generic sequence is genuinely Magidor generic and cofinal in the relevant measurable cardinal.

References

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