TWO QUESTIONS ON INFINITE-TIME COMPUTABLE EQUIVALENCE RELATIONS

JC

ABSTRACT. We answer two questions of Coskey and Hamkins concerning infinite-time computable reducibility. First, there are countable Borel equivalence relations which are strictly ordered by Borel reducibility but are bireducible by infinite-time computable functions. The proof combines an infinite-time uniformization theorem for analytic relations with a construction of Adams and Kechris. Second, the relation E_0 of equality modulo finite differences continuously embeds into both ordinary and eventual infinite-time degree equivalence, and the same holds relative to every real parameter. The second result follows from a category theorem: if a countable equivalence relation on Cantor space has the Baire property and contains E_0, then E_0 admits a continuous injective reduction to it.

1. INTRODUCTION

Coskey and Hamkins initiated a systematic comparison between Borel reducibility and reducibility by infinite-time Turing machines in [3]. Two of their questions ask whether the new reducibility genuinely changes the comparison of Borel equivalence relations, and how the basic nonsmooth countable Borel relation E_0 sits inside the infinite-time degree structures.

Their Question 3.5 asks whether there are Borel equivalence relations E, F such that

$$E \leq_c F \qquad \text{but} \qquad E \not\leq_B F,$$

where a fixed real parameter is permitted in the infinite-time computation. Their Question 4.9 asks whether E_0 reduces, in any reasonable sense, to either ordinary or eventual infinite-time degree equivalence. We prove the following.

Theorem 1.1. *(1) There are countable Borel equivalence relations E, F such that*

$$E <_B F \qquad \textit{and} \qquad E \sim_c F.$$

In particular, $F \leq_c E$ but $F \not\leq_B E$.

(2) For every real z, there is a continuous injection $f_z : 2^\omega \to 2^\omega$ which is simultaneously a reduction

$$E_0 \longrightarrow \equiv^z_\infty \qquad \textit{and} \qquad E_0 \longrightarrow \equiv^z_{e\infty} .$$

The two parts use different ideas. For the first, we prove an effective choice theorem for infinite-time machines: an analytic relation all of whose sections are nonempty has an infinite-time computable uniformization from a code for the relation. The analytic hypothesis cannot simply be replaced by parameter-free infinite-time decidability. By the No Uniformization Theorem of Hamkins and Lewis [4, Theorem 4.8], there is an infinite-time decidable relation with nonempty sections which has no infinite-time computable uniformization. Adams and Kechris constructed countable Borel equivalence relations E^* and E_T, indexed by a tree T, such that $E^* \leq_B E_T$ exactly when the closed set $[T]$ has a full Borel uniformization. Choosing T with full projection but without a Borel uniformization, the infinite-time choice theorem supplies the missing reduction while the Adams–Kechris argument rules out a Borel one.

For the second part, we use a Baire-category argument. A countable equivalence relation with the Baire property is meager as a subset of the square. We prove a block-fusion lemma which, for any comeager $C \subseteq (2^\omega)^2$, produces a continuous injection preserving finite differences and sending every pair with infinitely many differences into C. It follows that whenever a countable equivalence relation E has the Baire property and contains E_0, there is a continuous injective reduction of E_0 to E. Both infinite-time degree relations satisfy these hypotheses.

Section 2 fixes notation. Section 3 proves analytic uniformization and answers Question 3.5. Sections 4 and 5 establish the category theorem and answer Question 4.9.

2. Preliminaries

We work on Cantor space 2^ω and Baire space ω^ω, using standard recursive codings when a product or a sequence of reals is treated as a single real. For $x, y \in 2^\omega$,

$$x \; E_0 \; y \quad \iff \quad \exists n \, \forall m \geq n \, (x(m) = y(m)).$$

If E and F are equivalence relations on standard Borel spaces, then $E \leq_B F$ means that there is a Borel function f satisfying

$$x \; E \; y \quad \iff \quad f(x) \; F \; f(y).$$

Following [3], $E \leq_c F$ means that such a reduction is computed by an infinite-time Turing machine, with a fixed real parameter allowed. We write $E \sim_c F$ for mutual infinite-time computable reducibility. Every Borel function between recursively presented Polish spaces is infinite-time computable from a real code for the function [3, Theorem 2.1]. Thus a continuous or Borel reduction is automatically an infinite-time computable reduction in this sense.

For reals x, y, z, write

$$x \leq_\infty{}^z \; y$$

when an infinite-time Turing machine with oracle $y \oplus z$ halts with output x, and write

$$x \leq_{e\infty}{}^z \; y$$

when the output tape of such a computation eventually stabilizes to x. Mutual reducibility gives $\equiv_\infty^z$ and $\equiv_{e\infty}^z$. We omit the superscript when $z = 0$. Every equivalence class of either relation is countable, since, from a fixed oracle, each program has at most one halting output and at most one stabilized output.

We use the original Hamkins–Lewis ITTM convention, with the limsup rule for tape cells and the head returned to the first cell at limit stages. Equivalent conventions give the same computable functions. The synchronization argument in Section 3 is stated for this convention.

We shall also use the following regularity fact. Infinite-time semidecidable and semieventually decidable sets are absolutely Δ_2^1 [3, Proposition 2.4], and absolutely Δ_2^1 sets have the Baire property [2, Section 3.1]These statements relativize to a real parameter.

We occasionally pass between ω^ω and 2^ω. A Borel isomorphism and its inverse are infinite-time computable from suitable real codes, so transport along such an isomorphism preserves both Borel and infinite-time computable reducibility.

3. Analytic uniformization by infinite-time machines

The first ingredient is a direct consequence of the infinite-time decidability of well-foundedness. We give the direct infinite-time Turing machine construction needed here. The proof includes a synchronization device which ensures that the countable branch construction actually halts.

Theorem 3.1. *Let $A \subseteq \omega^\omega \times \omega^\omega$ be analytic, and let p code a tree presentation of A. If every vertical section A_x is nonempty, then there is a total function $u : \omega^\omega \to \omega^\omega$, infinite-time computable from p, such that*

$$(x, u(x)) \in A$$

for every $x \in \omega^\omega$.

Proof. Choose a tree

$$T \subseteq \omega^{<\omega} \times \omega^{<\omega} \times \omega^{<\omega}$$

recursive in p such that

$$(x, y) \in A \iff \exists w \in \omega^\omega\ (x, y, w) \in [T].$$

Fix x. Let T_x be the tree on pairs of finite sequences obtained by substituting x in the first coordinate. By hypothesis T_x is ill-founded.

An infinite-time Turing machine can decide whether a real codes a well-founded tree [4, Corollary 2.3]. We use this decision procedure to construct a branch through T_x. From p, x, and a finite node t, a code for the subtree $(T_x)_t$ is obtained uniformly. Suppose that a finite node s of T_x has already been selected and that the subtree above s is ill-founded. Examine the immediate extensions of s in a fixed recursive order. For each candidate t, run the well-foundedness decision procedure on $(T_x)_t$. Choose the first candidate for which this subtree is ill-founded. Such a candidate exists, and the search terminates because only finitely many candidates precede the first successful one and each well-foundedness test halts. Repeating this operation produces a branch

$$(y, w) \in [T_x].$$

The machine keeps the successive coordinates of w on a scratch tape and writes the successive coordinates of y permanently on its output tape.

It remains to arrange that the machine halts after all coordinates have been written. Reserve the first cell of a scratch tape as a pulse cell; the head sees this cell at every limit stage. The cell is normally 0, and the well-foundedness subroutines do not use it. Immediately after one further coordinate of the branch has been completed, set the pulse cell to 1 for one step and then reset it to 0. Let τ_n be the stage of the n-th pulse and let

$$\tau = \sup_{n\in\omega} \tau_n.$$

The pulse stages are strictly increasing. If $\delta < \tau$, then $\delta < \tau_n$ for some n, so only finitely many pulses occur below δ; consequently the limsup value of the pulse cell at every proper limit stage below τ is 0. At the limit stage τ, the pulses are cofinal, and the limsup value is 1. The transition out of the distinguished limit state checks the pulse cell: when its value is 0, the machine resumes the current coded simulation, and when its value is 1, it enters the halt state at the next step. By stage τ, every coordinate of y has been permanently written. Defining $u(x) = y$ completes the construction. □

Corollary 3.2. *Every nonempty analytic set contains a real infinite-time computable from a code for the set.*

Proof. If $B \subseteq \omega^\omega$ is nonempty and analytic, apply Theorem 3.1 to $A = \omega^\omega \times B$. □

We now apply the theorem to the Adams–Kechris construction. We recall only the part needed here. Adams and Kechris define a fixed countable Borel equivalence relation E^* and, for each tree T on $\omega \times \omega$, a countable Borel equivalence relation E_T. Their proof gives $E_T \leq_B E^*$ for every T, and their Lemma 5.2 shows that

$$E^* \leq_B E_T \iff [T] \text{ has a full Borel uniformization.}$$

Moreover, if $f : \omega^\omega \to \omega^\omega$ uniformizes $[T]$, then, in their coding of the two relations,

$$g(x, y) = (x, f(x), y)$$

is a reduction of E^* to E_T [1, Lemma 5.2]. The verification that g is a reduction uses only that $(x, f(x)) \in [T]$, not that f is Borel. They also prove that there is a tree T for which $[T]$ has full projection but no full Borel uniformization [1, discussion preceding Theorem 5.5].

Theorem 3.3. *There are countable Borel equivalence relations E, F such that*

$$E <_B F \qquad \textit{but} \qquad E \sim_c F.$$

Consequently, Question 3.5 of [3] *has a positive answer.*

Proof. Choose a tree T for which $[T]$ has full projection but no full Borel uniformization. By the Adams–Kechris facts above,

$$E_T \leq_B E^* \qquad \text{and} \qquad E^* \not\leq_B E_T.$$

Since every section of the closed relation $[T]$ is nonempty, Theorem 3.1 gives an infinite-time computable uniformizer f, using a code for T as a fixed real parameter. The displayed map $g(x,y) = (x, f(x), y)$ is therefore an infinite-time computable reduction of E^* to E_T. The Borel reduction in the other direction is also infinite-time computable from a code. Thus

$$E_T <_B E^* \qquad \text{and} \qquad E_T \sim_c E^*.$$

Transporting both relations along fixed Borel isomorphisms places them on Cantor space without changing either conclusion. ☐

Remark 3.4. The real coding T is used as a fixed parameter, exactly as allowed in Question 3.5. No parameter-free version of Theorem 3.3 is asserted. Theorem 3.1 is uniform relative to a tree code; the corresponding assertion for arbitrary decidable relations already fails without parameters by the no-uniformization theorem of Hamkins and Lewis.

4. A category lemma for E_0

The proof of the second part of Theorem 1.1 rests on two elementary category facts. The first is an immediate consequence of the Kuratowski–Ulam theorem.

Lemma 4.1. *Let $R \subseteq X \times Y$ have the Baire property, where X, Y are perfect Polish spaces. If every vertical section R_x is countable, then R is meager.*

Proof. Suppose that R is nonmeager. Since R has the Baire property, there are nonempty open sets $U \subseteq X$ and $V \subseteq Y$ such that R is comeager in $U \times V$. By the Kuratowski–Ulam theorem, for comeager many $x \in U$, the section R_x is comeager in V. This is impossible because every countable subset of a perfect Polish space is meager. ☐

The next lemma uses a Mycielski-style argument.

Lemma 4.2. *Let $C \subseteq (2^\omega)^2$ be comeager. There is a continuous injection $f : 2^\omega \to 2^\omega$ such that*

$$\begin{aligned} x \mathrel{E_0} y \quad &\Longrightarrow \quad f(x) \mathrel{E_0} f(y), \\ x \mathrel{\not\!E_0} y \quad &\Longrightarrow \quad (f(x), f(y)) \in C. \end{aligned}$$

Proof. Choose dense open sets $D_n \subseteq (2^\omega)^2$ such that

$$\bigcap_{n\in\omega} D_n \subseteq C.$$

We construct, for every n, two distinct finite binary blocks a_n^0, a_n^1 of equal length. Once the blocks below n have been chosen, let

$$P_n = \left\{ a_0^{i_0} \frown \cdots \frown a_{n-1}^{i_{n-1}} : (i_0, \ldots, i_{n-1}) \in 2^n \right\}.$$

We require that for all $u, v \in P_n$, all $k \leq n$, and all $i \neq j$ in $\{0, 1\}$,

$$[\, u \frown a_n^i \,] \times [\, v \frown a_n^j \,] \subseteq D_k. \tag{1}$$

There are only finitely many requirements at stage n. Begin with two empty temporary blocks b^0, b^1, and process the requirements one at a time. For a requirement involving u, v, i, j, k, density of D_k gives strings

$$s \supseteq u \frown b^i, \qquad t \supseteq v \frown b^j$$

with $[s] \times [t] \subseteq D_k$. Write $s = u \frown \widetilde{b}^i$ and $t = v \frown \widetilde{b}^j$, and replace b^i, b^j by $\widetilde{b}^i, \widetilde{b}^j$. Because D_k is open, all requirements already achieved remain true after further extensions. After the finite list has been processed, pad the two temporary blocks to the same length and append different final bits. The resulting blocks a_n^0, a_n^1 are distinct, have equal length, and satisfy (1).

Define

$$f(x) = a_0^{x(0)} \frown a_1^{x(1)} \frown a_2^{x(2)} \frown \cdots .$$

The equal-length condition makes f continuous, and the blocks being distinct makes it injective. If $x E_0 y$, then only finitely many corresponding blocks differ, so $f(x) E_0 f(y)$.

Suppose $x \not\!\!E_0 y$. Given k, choose $n \geq k$ with $x(n) \neq y(n)$. The output prefixes before the n-th block belong to P_n, and the stage-n requirement yields $(f(x), f(y)) \in D_k$. Since this holds for every k, the pair belongs to C. □

Theorem 4.3. *Let E be an equivalence relation on 2^ω with the Baire property. Suppose that every E-class is countable and*

$$E_0 \subseteq E.$$

Then there is a continuous injective reduction f of E_0 to E. Moreover, f may be chosen so that

$$x E_0 y \quad \Longrightarrow \quad f(x) E_0 f(y).$$

Proof. By Lemma 4.1, E is meager in $(2^\omega)^2$. Apply Lemma 4.2 to the comeager set $(2^\omega)^2 \setminus E$, obtaining f. If $x E_0 y$, then $f(x) E_0 f(y)$, and hence $f(x) E f(y)$ because $E_0 \subseteq E$. If $x \not\!\!E_0 y$, then $(f(x), f(y)) \notin E$. Thus f is the required reduction. □

Remark 4.4. The equivalence-relation hypothesis is used only in the final application. The same proof applies to any relation with the Baire property and countable vertical sections which contains E_0.

5. Infinite-time degree equivalence

We apply Theorem 4.3 to the two degree relations.

Proposition 5.1. *For every real z, the relations $\equiv^z_\infty$ and $\equiv^z_{e\infty}$ have the Baire property, all of their classes are countable, and*

$$E_0 \subseteq \equiv^z_\infty \subseteq \equiv^z_{e\infty} .$$

Proof. The ordinary relation $\equiv^z_\infty$ is infinite-time semidecidable relative to z, by the usual dovetailing argument for the two reducibilities [4, Theorem 5.7]. The eventual relation $\equiv^z_{e\infty}$ is infinite-time eventually decidable relative to z [3, Proposition 4.6]. Hence both relations have the Baire property by the regularity facts recalled in Section 2. Each class is countable because there are only countably many programs. Finally, finite modifications are ordinarily Turing equivalent, which gives the displayed inclusions. □

Theorem 5.2. *For every real z, there is a continuous injection $f_z : 2^\omega \to 2^\omega$ such that, for all $x, y \in 2^\omega$,*

$$xE_0y \iff f_z(x) \equiv^z_\infty f_z(y) \iff f_z(x) \equiv^z_{e\infty} f_z(y).$$

In particular, E_0 continuously embeds into both infinite-time degree equivalence relations.

Proof. Apply Theorem 4.3 to $E = \equiv^z_{e\infty}$, choosing the reduction with the additional property stated there. The resulting map f_z reduces E_0 to eventual degree equivalence. If xE_0y, then $f_z(x)E_0f_z(y)$, so the two images are already ordinarily Turing equivalent and therefore $\equiv^z_\infty$-equivalent. If $x \not\!E_0y$, then the images are not $\equiv^z_{e\infty}$-equivalent, and hence cannot be $\equiv^z_\infty$-equivalent. Thus the same map reduces E_0 to both relations. □

Taking $z = 0$ gives a positive answer to Question 4.9 of [3], and in a stronger form than Borel reducibility: the reductions are continuous injections.

References

[1] S. Adams and A. S. Kechris, *Linear algebraic groups and countable Borel equivalence relations*, J. Amer. Math. Soc. **13** (2000), no. 4, 909–943.

[2] M. Carl and P. Schlicht, *Randomness via infinite computation and effective descriptive set theory*, J. Symbolic Logic **83** (2018), no. 2, 766–789.

[3] S. Coskey and J. D. Hamkins, *Infinite time decidable equivalence relation theory*, Notre Dame J. Formal Logic **52** (2011), no. 2, 203–228.

[4] J. D. Hamkins and A. Lewis, *Infinite time Turing machines*, J. Symbolic Logic **65** (2000), no. 2, 567–604.