A belated foundational role of set theory

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2025/07/27
2025 Annual Conference of Philosophy of Mathematics in China

TL;DR

Takehome message

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There is a role that set theory plays in the foundations of mathematics, which is best characterized as **Brickwalling**.

- One finds bits of it sprinkled across the foundational discourse, especially in relation to model theory and category theory.
- But it itself has not been properly addressed in the relevant literature yet, partly because it is only revealed after we trace a certain thread of development in descriptive set theory (DST).
- And this peculiar nature invites questions about the ways in which Brickwalling is foundational, and how set theory plays that role.

Agenda

Takehome message

There is a role that set theory plays in the foundations of mathematics, which is best characterized as **Brickwalling**.

- What do we want a foundation to do? A second-philosophical[1] take on the foundations of mathematics
- Crash course on DST history - and how Brickwalling came about
- Implications for what it means to be foundational and what it means for a theory to play a foundational role

[1] In the sense of (Maddy, 2007)

Foundations

What do we want a foundation to do?

Our method of analysis (Maddy, 2017, 2019)

"I'm skeptical of the **unspoken assumption that there's an underlying concept of a 'foundation' up for analysis**, that this analysis would properly guide our assessment of the various candidates. In contrast, it seems to me that the considerations the combatants offer against opponents and for their preferred candidates, as well as the roles each candidate actually or potentially succeeds in playing, reveal quite a number of different jobs that mathematicians want done. What matters is these **jobs we want our theories to do** and **how well they do them**."

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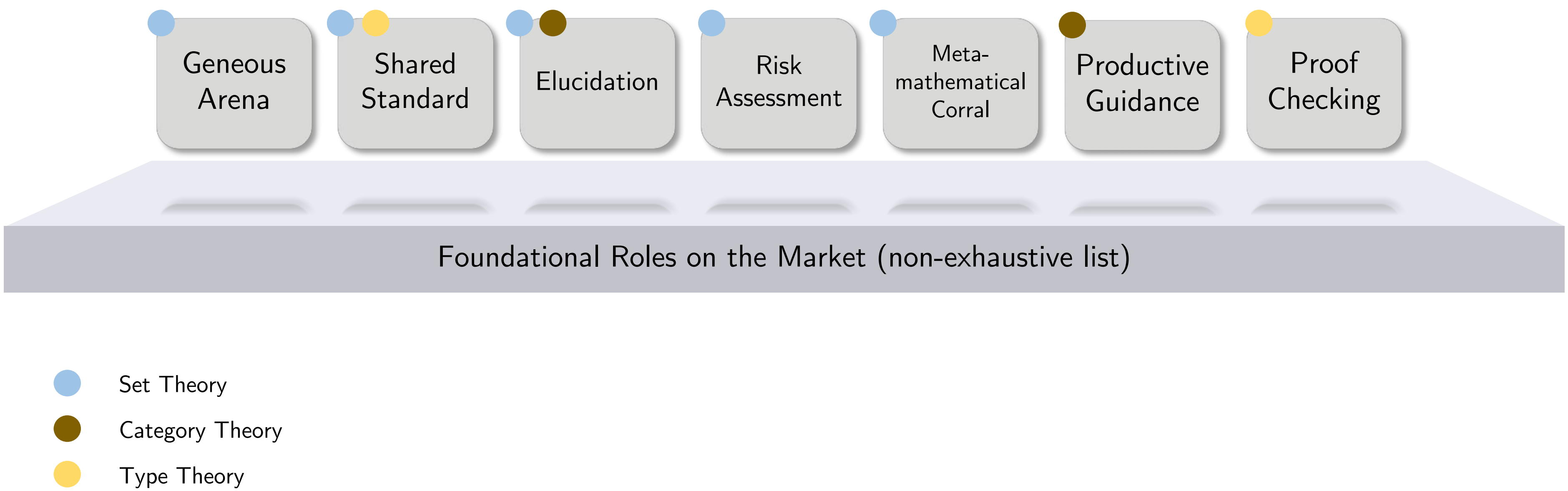
So, a disclaimer

The contribution here is not to help set theory get ahead in some sort of foundational race, but rather to faithfully capture a specific line of mathematical practice.

Maddy on Foundation



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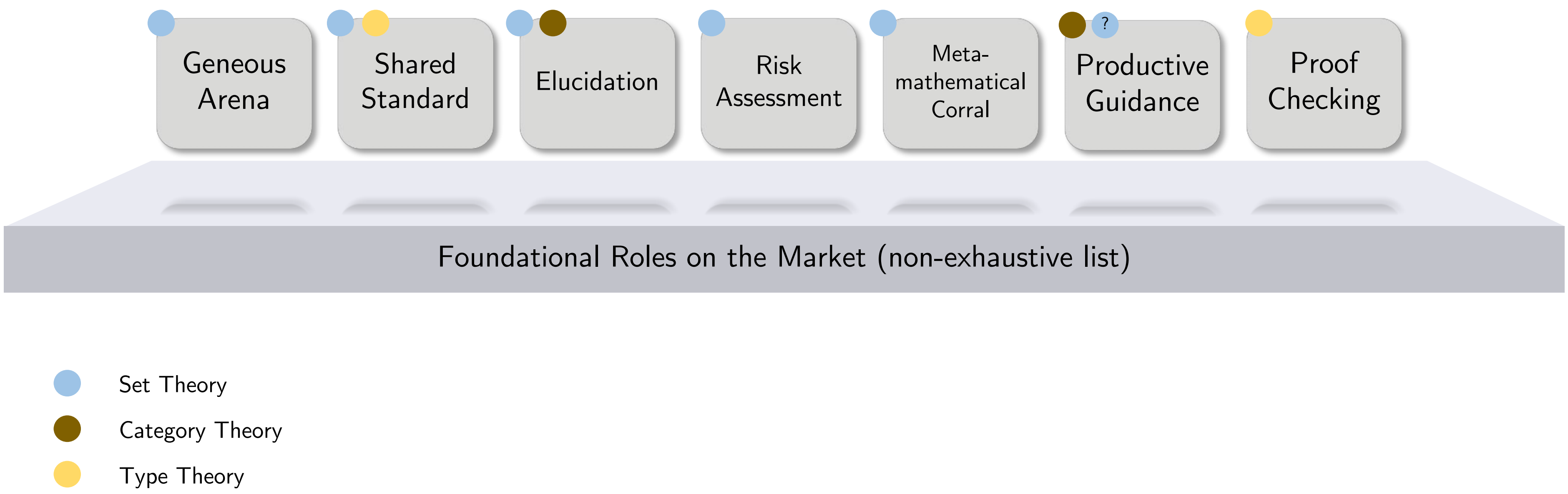


Maddy on Foundation

Set Theory

Category Theory

Type Theory

Productive Guidance

Maddy on category theory

"What category theory has accomplished...is a way of **thinking** about a large part of mathematics, of **organizing** and **understanding** it, that's been immensely **fruitful in practice**. "

Definition: In the context of cyclic diagrams, co-Kan co-homologies are simply internal units over resolutions over the arrow of hom-objects.

- A lift of a monoidal $\mathfrak{K}\backslash w$-proper object classifier $\mathcal{F}$ P augmented with a $\mathcal{A} \leftarrow E^{\mathcal{E}([\mathbf{b}^k])}$-infinite ${\mathscr{V}_{\hat{\mathbf{Q}}}}^{\mathscr{Q}}$-indexed N-complete tensor $\mathcal{F}$ augmented with a internally $[\mathfrak{S}\backslash\mathbf{Q} \to x]$-co-closed hom-object $\mathbf{U}$ satisfies the right property if composable covers are "structure-preserving" from X's point of view
- All global products are simplicial in the following manner:

$$\bigoplus_p \mathbf{k} \to L = \left[\bigcup_s^{\mathfrak{D}} O \Rightarrow R\right]^{\mathfrak{t} \to \mathfrak{W}}$$

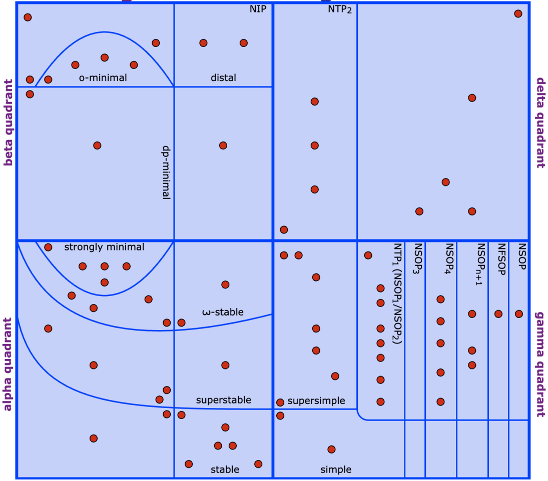
Productive Guidance

Baldwin on model theory (Baldwin, 2024)

"Shelah's Classification theory divides complete first order theories by syntactical conditions into a small number of classes. Theories in the same class share mathematically significant properties ... enabling the **transfer** of results from one theory to another in the same class and provides **guidance** to distinguish the wild from the tame."

forking and dividing

Map of the Universe



Nice Properties of Theories

ω-stable	superstable	stable	
strongly minimal	dp-minimal	o-minimal	
supersimple	simple	NIP	distal
NTP$_1$ (NSOP$_1$/NSOP$_2$)	NTP$_2$	NSOP	
NSOP$_3$	NSOP$_4$	NSOP$_{n+1}$	NFSOP

Click a property above to highlight region and display details. Or click the map for specific region information.

Reset

List of Examples

- ACF
- $\mathbb{Q}$-vector spaces
- $(\mathbb{Z},\ x \mapsto x+1)$
- Hrushovski's new strongly minimal set
- infinite sets
- everywhere infinite forest
- infinitely expanding equivalence relations
- Farey graph
- $((\mathbb{Z}/4\mathbb{Z})^{\omega}, +)$
- DCF$_0$

Implications Between Properties

Open Regions

Open Examples

Questions? Suggestions? Corrections? email me: gconant@uic.edu

References Update Log

supported by the NSF under grant no. DMS-2204787

What about set theory?

A tension

Set theory, in its capacity to provide a Generous Arena, to perform Mathematical Corral, etc, carries tension with providing Productive Guidance.

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Maddy, 2019

"Unfortunately, [Productive] Guidance is in serious tension with Generous Arena and Shared Standard; long experience suggests that ways of thinking beneficial in one area of mathematics are unlikely to be beneficial in all areas of mathematics."

Baldwin's Distinction

Figure: Scaffold

Figure: Foundation

Baldwin's Distinction

Figure: Scaffold

Figure: Foundation

Baldwin's Distinction

1. Consistency
2. Interpretations
3. Metamathematics
4. etc

Figure: Foundation

Baldwin's Distinction

Figure: Scaffold

1. Local Foundation
2. Unity
3. Productive Guidance

Story of Brickwalling

Back in 1898...

Borel, 1898: When in doubt, rule out

Faced with the proliferation of intractable sets of real numbers, Borel opted for a context-restriction strategy...

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Definition

A *Borel set* is a set that can be constructed from open sets in the real line by countable unions, intersections, and complements.

Origins of the Borel sets

...[other sets not satisfying these properties] will be useless to us, even hinder us...

Les ensembles dont on peut définir la mesure en vertu des définitions précédentes seront dits par nous ensembles mesurables, sans que nous entendions impliquer par là qu'il n'est pas possible de donner une définition de la mesure d'autres ensembles; mais une telle définition nous serait inutile; elle pourrait même nous gêner, si elle ne laissait pas à la *mesure* les propriétés fondamentales que nous lui avons attribuées dans les définitions que nous avons données (1).

Ces propriétés essentielles, que nous résumons ici parce qu'elles nous seront utiles, sont les suivantes : La mesure de la somme d'une infinité dénombrable d'ensembles est égale à la somme de leurs mesures; la mesure de la différence de deux ensembles est égale à la différence de leurs mesures (2); *la mesure n'est jamais négative; tout ensemble dont la mesure n'est pas nulle n'est pas dénombrable.* C'est surtout de cette dernière propriété que nous ferons usage. Il est d'ailleurs expressément entendu que nous ne parlerons de mesure qu'à propos des ensembles que nous avons appelés *mesurables*.

...it is expressly understood that we will speak of measure only in connection with the sets that we have called measurable.

Borel (1898). Leçons sur la théorie des fonctions.

Passing the torch to Moscow

Luzin's Program

To extend the structural analysis of the Borel sets (by e.g., Borel, Lebesgue, and Baire) to the projective sets.

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Definition

A set $X \subseteq \mathbb{R}$ is projective iff it can be obtained from open sets by finitely many applications of complementation and continuous images. (Fact: this includes the Borel sets.)

...a **finite law** that defines a (choice) set of points...

Il en est tout autrement pour le cas singulier où nous pouvons tirer de R une loi finie λ qui définit un ensemble de points L ouissant des deux propriétés suivantes:

1° xRx' est fausse si les points x et x' ($x \neq x'$) appartiennent à L;

2° Quel que soit le point y pris dans le continu, il existe un point x de L tel que xRy est vraie.

Nous appellerons *partage lebesguien* tout partage qui possède ces deux propriétés. C'est dans ce cas seul que la totalité T existe réellement, étant achevée; elle est donc légitime.

Mais, dans le cas général où nous n'avons plus du partage lebesguien, la totalité T est, à notre avis, tout illégitime: *ce n'est qu'une pure virtualité.*

Luzin(1927). Sur les ensembles analytiques.

Luzin on legitimacy

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*We will call any partition that possesses these two properties a Lebesgue partition. It is only in this case that the totality T **truly exists**, being complete; it is therefore **legitimate**.*

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Mais, dans le cas général où nous n'avons plus du partage lebesguien, la totalité T est, à notre avis, tout illégitime: *ce n'est qu'une pure virtualité.*

*...where we no longer have a Lebesgue partition, the totality T is...totally **illegitimate**...*

Luzin(1927). Sur les ensembles analytiques.

Hitting a brickwall

Luzin (1925), *Les propriétés des ensembles projectifs*

One does not know, and one will never know, whether the PCA ($\underset{\sim}{\Sigma}^1_2$) sets are Lebesgue measurable.

We now know why:

Independence!

Similar brickwalls

Farah, ICM 2014

"The representation theory of nonseparable algebras was largely abandoned because some of the central problems proved to be intractable"

Brickwalls from unprovability

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A criticism

Isn't this just Risk Assessment?

Claim

While the previous examples have distinct flavors of Risk Assessment, set theory provides yet another kind of Brickwalling...

An example: von Neumann's isomorphism problem

In 1932, von Neumann essentially set out to classify measure-preserving transformations: how do we determine whether two measure-preserving transformations are isomorphic?

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morphieinvarianten Eigenschaften. Vermutlich kann sogar zu jeder allgemeinen Strömung eine isomorphe stetige Strömung gefunden werden[13], vielleicht sogar eine stetig-differentiierbare, oder gar eine mechanische.

[13] Der Verfasser hofft, hierfür demnächst einen Beweis anzugeben.

Ornstein's Classification Theorem (1970)

Definition

A Bernoulli shift is a quadruple $(X, \mathcal{B}, \mu, T)$ such that

1. $X = \{1, 2, .., n\}^{\mathbb{Z}}$ for some natural number n
2. $\mathcal{B}$ is the Borel σ-algebra on X
3. μ is a product measure given by a probability distribution $(p_1, ..., p_n)$ with $\sum p_i = 1$
4. T shifts the space: for $x = (x_n)_{n \in \mathbb{Z}}$, $(Tx)_n = x_{n-1}$

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Definition

The Kolmogorov-Sinai entropy of a Bernoulli shift is $-\sum_{i=1}^{n} p_i \log p_i$

Definition

Two Bernoulli shifts $(X, \mathcal{B}, \mu, T)$ and$(Y, \mathcal{C}, \nu, S)$ are isomorphic if there is a measure-preserving map Φ from a μ-measure 1 subset of X onto a ν-measure 1 subset of Y such that $\Phi(Tx) = S\Phi(x)$ for μ-a.e. $x \in X$.

Kolmogorov-Sinai entropy

Theorem (Kolmogorov-Sinai, 50s)

If two Bernoulli shifts are isomorphic, then they have the same KS entropy (which is a real number).

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Theorem (Ornstein 1970)

Two Bernoulli shifts are isomorphic if and only if they have the same entropy.

A classic case of Borel reduction

Point: for each Bernoulli shift, we associate (in a Borel way) a real number, i.e., its entropy, such that the problem of isomorphism is completely reduced to the problem of identity.

Template

Borel map F : Bernoulli shifts $\to \mathbb{R}$, such that

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Template

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We now say the Bernoulli shifts are completely *classified* by their entropy.

Invariant descriptive set theory

Results like this gave rise to

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Borel equivalence relations theory

Abstract study of the hierarchy of classification problems.

Definition

Let E_1, E_2 be a Borel equivalence relation on standard Borel spaces X_1, X_2, respectively. We say E_1 is *Borel reducible* to E_2 (written as $E_1 \leq_B E_2$) iff there is a Borel function $F : X_1 \rightarrow X_2$ such that $uE_1v \Leftrightarrow f(u)E_2f(v)$. Such a function F is called a *Borel reduction* of E_1 to E_2.

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Borel equivalence relations theory

Abstract study of the hierarchy of classification problems.

Intuition

A Borel reduction $F : (X_1, E_1) \to (X_2, E_2)$ associates, in a reasonably concrete way, each $x \in X_1$ with a **complete invariant** $y \in X_2$. This way, to know whether $u, v \in X_1$ fall in the same classification, we can just check if they get assigned equivalent invariants. In a number of familiar cases, E_2 is just Identity on some Polish space.

First result of Borel equivalence relations theory

...we will have that φ(x)=φ(x') for x−x' rational and φ(x)≠φ(x') for x−x' irrational

Soit maintenant x un nombre réel donné. Designons par $E(x)$ l'ensemble de tous les nombres $x+r$, r étant un nombre rationnel quelconque: on voit sans peine que ce sera un ensemble dénombrable et que nous aurons toujours $E(x) = E(x')$ pour $x - x'$ rationnel et $E(x) \neq E(x')$ pour $x - x'$ irrationnel.

A tout nombre réel donné x correspondra donc un nombre réel $\varphi(x) = f[E(x)]$, et il suit des propriétés de $E(x)$ et $f(E)$ que nous aurons $\varphi(x) = \varphi(x')$ pour $x - x'$ rationnel et $\varphi(x) \neq \varphi(x')$ pour $x - x'$ irrationnel.

Or, je dis que toute fonction $\varphi(x)$ jouissant de cette propriété est non mesurable (1).

I claim that any function having these properties is non-measurable

Sierpiński (1917), Sur quelques problèmes qui impliquent des fonctions non-mesurables.

Scaffolding by DST

Set theory provides a scaffold (in Baldwin's sense) a certain parts of mathematics - those that can be codified as definable equivalence relations on Polish spaces (many classification programs, for example).

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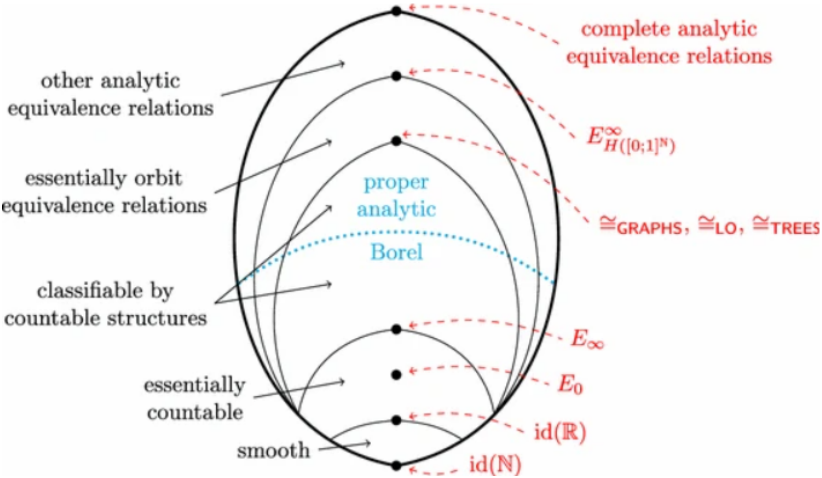

Borel strikes again...

Borel = Tractable

Ros, 2021

“As a matter of fact, the vast majority of classification problems naturally occurring in mathematics falls, up to a suitable coding procedure, inside this class [of analytic equivalence relations]. However, equivalence relations of this kind may be very complicated and intractable, so when possible it is customary to restrict the attention to the following strictly smaller class [of Borel equivalence relations].”

Generalized Church's Thesis

Foreman and Gorodetski, 2022

A problem can be solved with inherently countable techniques just in case it can be viewed as Borel in a Polish Space.

Theorem (Foreman et al., 2011, *Annals of Mathematics*)

The isomoprhism relation between ergodic measure-preserving transformations is not Borel (relative to natural, suitable background space).

Brickwalling: modern articulations

Foreman et al., 2011

"This result explains, perhaps, why the problem of determining whether ergodic transformations are isomorphic or not has proven to be so **intractable** ... [the theorem] can be interpreted as saying that there is no method or protocol that involves a countable amount of information and countable number of steps that reliably distinguishes between nonisomorphic ergodic measure preserving transformations. We view this as a **rigorous way of saying that the classification problem for ergodic measure preserving transformations is intractable.**"

Another example: Smale's Program

Smale, ICM 1962

Classify the diffeomorphisms of a manifold M by topological conjugacy. That is, find reductions of the equivalence relation $f \sim_{top} g$ for $f, g \in \mathrm{Diff}(M)$, where $f \sim_{top} g$ iff there is a homeomorphism $h : M \to M$ such that $h \circ f = g \circ h$.

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Theorem (Foreman, Gorodetski)

This equivalence relation is not Borel (relative to natural, suitable background space)

Foreman and Gorodetski, 2022

"The main result of this paper is that Smale's Program is hopeless in the rigorous sense"

Brickwalling

By delineating the boundary between the tractable and intractable, set theory has been able to produce a kind of guidance that is productive - that is, beneficial to mathematician's productivity - by pinpointing the fundamental obstacles with one's problems and/or methods.

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- Early on, this materialized in the form of revealing the unprovability of certain questions about projective sets.
- But it truly took form by way of the vast amount of anti-classification results.

Summary

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DST's venture into definable equivalence has come to form a scaffold for particular areas of mathematics, and this has provided a peculiar kind of guidance (perhaps negatively, yet still productive), which is best characterized as **Brickwalling**.

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- This foundational job is best appreciated in the hindsight of the modern development of Borel equivalence relations theory.

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- This sits in contrast to the kind of Productive Guidance that category theory and model theory provide, which is more about providing a framework for thinking about and organizing mathematics, ultimately assisting in producing new theorems.
- This foundational job is best appreciated in the hindsight of the modern development of Borel equivalence relations theory.
- It explains why certain research programs have been stalled or abandoned, by pinpointing the fundamental obstacles with one's problems and/or methods.

The End

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