The emergence of a foundational role

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TL;DR

Takehome Message

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- But it itself has not been properly addressed in the relevant literature yet, partly because it is only revealed after we trace a certain thread of development in descriptive set theory (DST).

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There is a role that set theory plays in the foundations of mathematics, which is best characterized as **Barrier Exposure**.

- One finds bits of it sprinkled across the foundational discourse, especially in relation to model theory and category theory.
- But it itself has not been properly addressed in the relevant literature yet, partly because it is only revealed after we trace a certain thread of development in descriptive set theory (DST).
- And this peculiar nature invites questions about the ways in which Barrier Exposure is foundational, and how set theory plays that role.

Agenda

Takehome Message

There is a role that set theory plays in the foundations of mathematics, which is best characterized as **Barrier Exposure**.

1. Background context: What do we want a foundation to do?
2. Crash course on DST history, and how Barrier Exposure came about
3. Continuing on to more modern developments

Foundations

What do we want a foundation to do?

Our method of analysis (Maddy, 2017, 2019)

"I'm skeptical of the **unspoken assumption that there's an underlying concept of a 'foundation' up for analysis**, that this analysis would properly guide our assessment of the various candidates. In contrast, it seems to me that the considerations the combatants offer against opponents and for their preferred candidates, as well as the roles each candidate actually or potentially succeeds in playing, reveal quite a number of different jobs that mathematicians want done. What matters is these **jobs we want our theories to do** and **how well they do them**."

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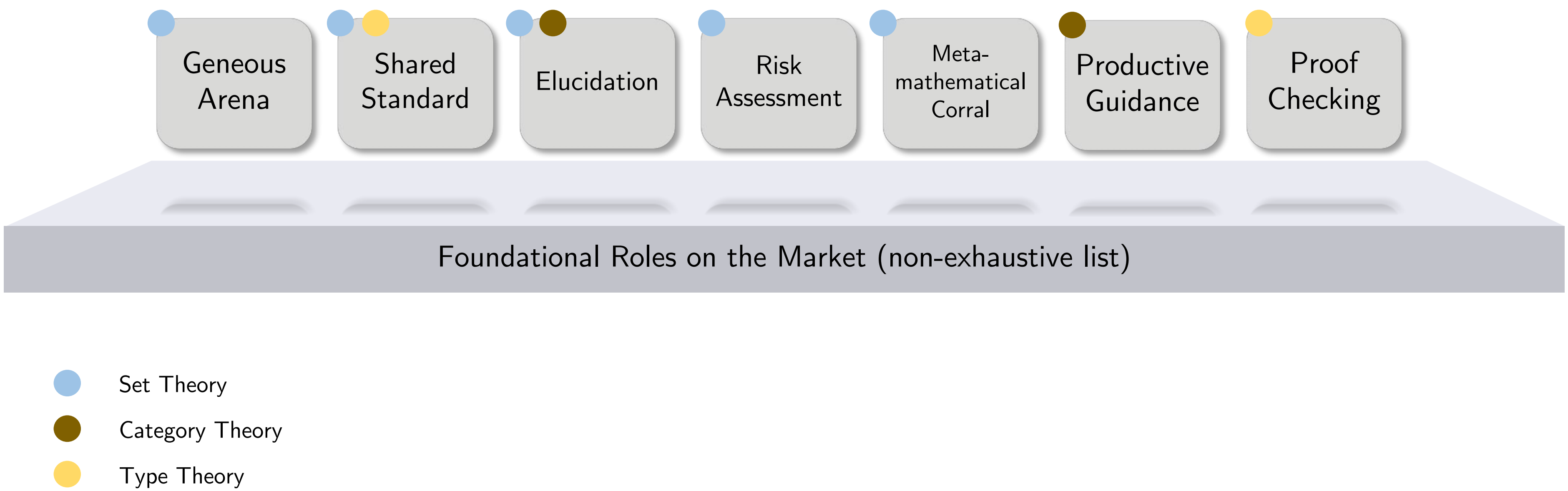
So, a disclaimer

The contribution here is not to help set theory get ahead in some sort of foundational race, but rather to faithfully capture a specific line of mathematical practice.

Maddy on Foundation



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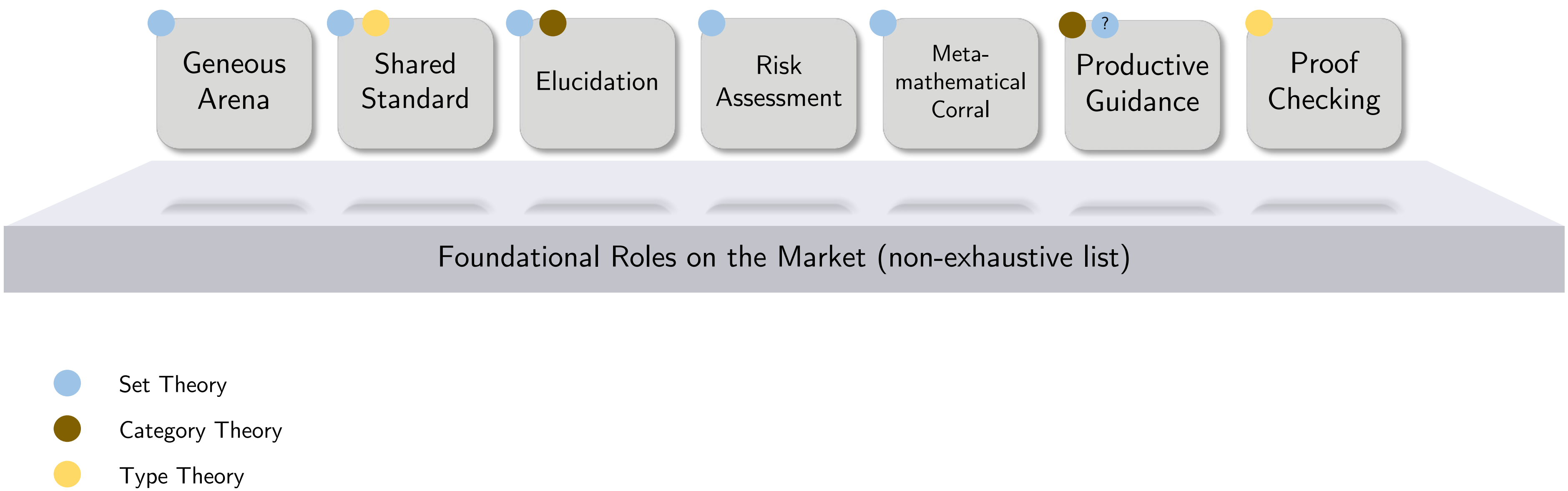

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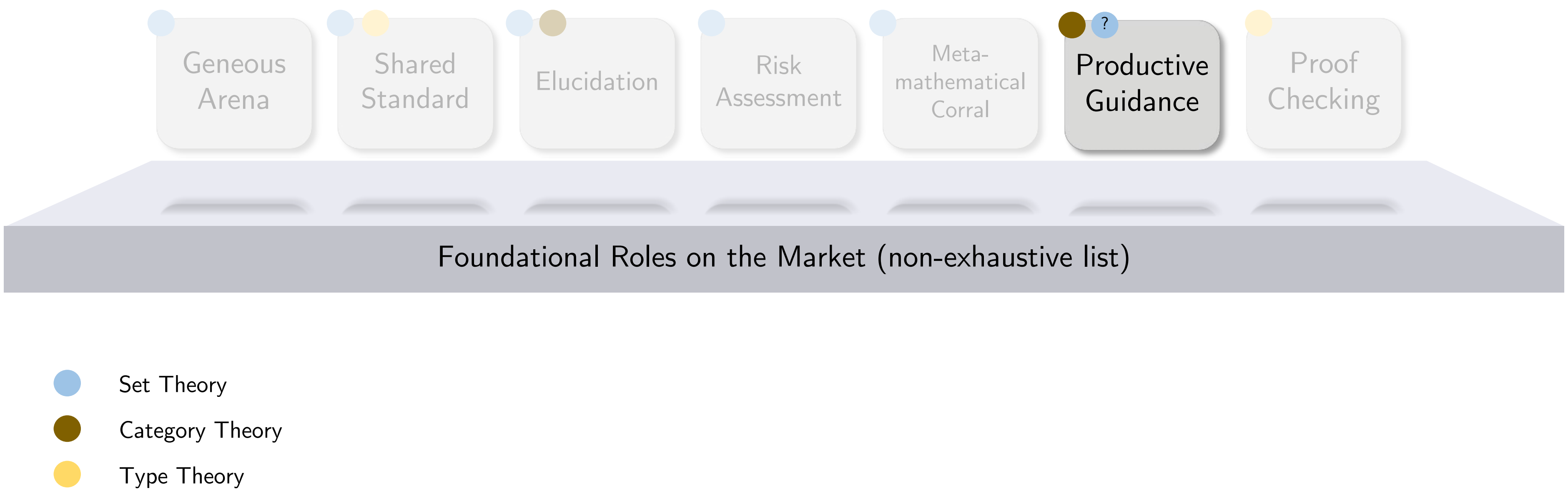


Productive Guidance

Maddy on category theory

"What category theory has accomplished...is a way of **thinking** about a large part of mathematics, of **organizing** and **understanding** it, that's been immensely **fruitful in practice**. "

Definition: In the context of cyclic diagrams, co-Kan co-homologies are simply internal units over resolutions over the arrow of hom-objects.

- A lift of a monoidal $\mathfrak{K}\backslash w$-proper object classifier $\mathcal{F}\ P$ augmented with a $\mathcal{A} \leftarrow E^{\mathcal{E}^{\left(\left[\mathbf{b}^k\right]\right)}}$-infinite ${\mathscr{V}_{\hat{\mathbf{Q}}}}^{\mathscr{Q}}$-indexed N-complete tensor $\mathcal{F}$ augmented with a internally $[\mathfrak{S}\backslash\mathbf{Q} \to x]$-co-closed hom-object $\mathbf{U}$ satisfies the right property if composable covers are "structure-preserving" from X's point of view
- All global products are simplicial in the following manner:

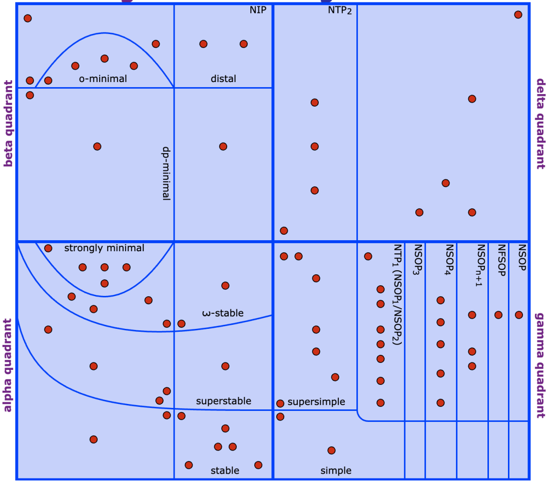
$$\bigoplus_p \mathbf{k} \to L = \left[\bigcup_s^{\mathfrak{D}} O \Rightarrow R\right]^{\mathrm{t} \to \mathfrak{W}}$$

Productive Guidance

Baldwin on model theory (Baldwin, 2024)

"Shelah's Classification theory divides complete first order theories by syntactical conditions into a small number of classes. Theories in the same class share mathematically significant properties … enabling the **transfer** of results from one theory to another in the same class and provides **guidance** to distinguish the wild from the tame."

forking and dividing



Questions? Suggestions? Corrections? email *me*: gconant@uic.edu

References **Update Log**

Map of the Universe

Nice Properties of Theories

ω-stable	superstable	stable	
strongly minimal	dp-minimal	o-minimal	
supersimple	simple	NIP	distal
NTP$_1$ (NSOP$_1$/NSOP$_2$)	NTP$_2$	NSOP	
NSOP$_3$	NSOP$_4$	NSOP$_{n+1}$	NFSOP

Click a property above to highlight region and display details. Or click the map for specific region information.

Reset

List of Examples

- ACF
- $\mathbb{Q}$-vector spaces
- $(\mathbb{Z}, x \mapsto x+1)$
- Hrushovski's new strongly minimal set
- infinite sets

- everywhere infinite forest
- infinitely expanding equivalence relations
- Farey graph
- $((\mathbb{Z}/4\mathbb{Z})^\omega, +)$

- DCF$_0$

Implications Between Properties

Open Regions

Open Examples

supported by the NSF under grant no. DMS-2204787

What about set theory?

A tension

Set theory, in its capacity to provide a Generous Arena, to perform Meta-mathematical Corral, etc, carries tension with providing Productive Guidance.

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Maddy, 2019

"Unfortunately, [Productive] Guidance is in serious tension with Generous Arena and Shared Standard; long experience suggests that ways of thinking beneficial in one area of mathematics are unlikely to be beneficial in all areas of mathematics."

Baldwin's Distinction

Figure: Scaffold

Figure: Foundation

Baldwin's Distinction

1. Consistency
2. Interpretations
3. Metamathematics
4. etc

Figure: Foundation

Baldwin's Distinction

Figure: Scaffold

1. Local Foundation
2. Unity
3. Productive Guidance

Story of Barrier Exposure

Back in 1898...

Borel, 1898: When in doubt, rule out

Faced with the proliferation of intractable sets of real numbers, Borel opted for a context-restriction strategy...

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Definition

A *Borel set* is a set that can be constructed from open sets in the real line by countable unions, intersections, and complements.

Origins of the Borel sets

...[other sets not satisfying these properties] will be useless to us, even hinder us...

Les ensembles dont on peut définir la mesure en vertu des définitions précédentes seront dits par nous ensembles mesurables, sans que nous entendions impliquer par là qu'il n'est pas possible de donner une définition de la mesure d'autres ensembles; mais une telle définition nous serait inutile; elle pourrait même nous gêner, si elle ne laissait pas à la *mesure* les propriétés fondamentales que nous lui avons attribuées dans les définitions que nous avons données (1).

Ces propriétés essentielles, que nous résumons ici parce qu'elles nous seront utiles, sont les suivantes : La mesure de la somme d'une infinité dénombrable d'ensembles est égale à la somme de leurs mesures; la mesure de la différence de deux ensembles est égale à la différence de leurs mesures (2); *la mesure n'est jamais négative; tout ensemble dont la mesure n'est pas nulle n'est pas dénombrable.* C'est surtout de cette dernière propriété que nous ferons usage. Il est d'ailleurs expressément entendu que nous ne parlerons de mesure qu'à propos des ensembles que nous avons appelés *mesurables*.

...it is expressly understood that we will speak of measure only in connection with the sets that we have called measurable.

Borel (1898). Leçons sur la théorie des fonctions.

Passing the torch to Moscow

Luzin's Program

To extend the structural analysis of the Borel sets (by e.g., Borel, Lebesgue, and Baire) to the projective sets.

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Definition

A set $X \subseteq \mathbb{R}$ is projective iff it can be obtained from open sets by finitely many applications of complementation and continuous images. (Fact: this includes the Borel sets.)

Luzin on legitimacy

...a **finite law** that defines a (choice) set of points...

Il en est tout autrement pour le cas singulier où nous pouvons tirer de R une loi finie λ qui définit un ensemble de points L ouissant des deux propriétés suivantes:

1° xRx' est fausse si les points x et x' $(x \neq x')$ appartiennent à L;

2° Quel que soit le point y pris dans le continu, il existe un point x de L tel que xRy est vraie.

Nous appellerons *partage lebesguien* tout partage qui possède ces deux propriétés. C'est dans ce cas seul que la totalité T existe réellement, étant achevée; elle est donc légitime.

Mais, dans le cas général où nous n'avons plus du partage lebesguien, la totalité T est, à notre avis, tout illégitime: *ce n'est qu'une pure virtualité.*

Luzin(1927). Sur les ensembles analytiques.

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*We will call any partition that possesses these two properties a Lebesgue partition. It is only in this case that the totality T **truly exists**, being complete; it is therefore **legitimate**.*

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Mais, dans le cas général où nous n'avons plus du partage lebesguien, la totalité T est, à notre avis, tout illégitime: *ce n'est qu'une pure virtualité.*

...where we no longer have a Lebesgue partition, the totality T is...totally ***illegitimate:*** *it's nothing but a pure virtuality*

Luzin(1927). Sur les ensembles analytiques.

Hitting a barrier

Luzin (1925), *Les propriétés des ensembles projectifs*

One does not know, and one will never know, whether the PCA ($\underset{\sim}{\Sigma}^1_2$) sets are Lebesgue measurable.

We now know why:

Independence!

Similar barriers

Farah, ICM 2014

"The representation theory of nonseparable algebras was largely abandoned because some of the central problems proved to be intractable"

Barriers from unprovability

Set theory provided insights into why the study of projective sets and nonseparable algebras hit barriers - fundamental questions are unprovable.

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A criticism

Isn't this just Risk Assessment?

Claim

While the previous examples have distinct flavors of Risk Assessment, set theory provides yet another kind of Barrier Exposure...

Two Programs

Sierpiński's Program

Sierpiński and the Axiom of Choice

- Earlier debates about the axiom of choice tended to proceed on philosophical grounds. As evidenced in the *Cinq Lettres* (Hadamard, 1905).

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The Shift in Sierpiński, 1965

It is most desirable to distinguish between theorems which can be proved without the aid of the axiom of choice and those which we are not able to prove without the aid of this axiom. Analysing proofs based on the axiom of choice we can

1. *Ascertain that the proof in question makes use of the axiom of choice.*
2. *Determine that the axiom of choice is sufficient for the proof of the theorem in question.*
3. *Determine that the axiom of choice is necessary for the proof of the theorem in question.*

Sierpiński and the Axiom of Choice

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The Shift in Sierpiński, 1965

Mathematical Existence $\rightsquigarrow$ Mathematical Productivity

An instance of Sierpiński's Program

The goal of this note is to call attention to problems ... that imply the existence of non-measurable functions.

Le but de cette Note est d'appeler l'attention sur quelques autres problèmes de la théorie des ensembles et certaines questions d'analyse qui impliquent l'existence des fonctions non mesurables.

On regarde dans la théorie des ensembles comme bien démontré que l'ensemble de tous les sous-ensembles dénombrables du continu a la puissance du continu. Or, nous allons démontrer que ce problème implique des fonctions non mesurables.

We consider well-established ... that the collection of countable subsets of the continuum has the power of the continuum. We are going to show that this implies non-measurable functions exist

Sierpiński (1917), Sur quelques problèmes qui impliquent des fonctions non-mesurables.

von Neumann's Program

Motivation: von Neumann's isomorphism problem

In 1932, von Neumann essentially set out to classify measure-preserving transformations: how do we determine whether two measure-preserving transformations are isomorphic?

...a continuous stream can be found for each general stream... or even a mechanical one

morphieinvarianten Eigenschaften. Vermutlich kann sogar zu jeder allgemeinen Strömung eine isomorphe stetige Strömung gefunden werden[13], vielleicht sogar eine stetig-differentiierbare, oder gar eine mechanische.

[13] Der Verfasser hofft, hierfür demnächst einen Beweis anzugeben.

The author hopes to provide a proof of this shortly.

J. v. Neumann (1932). Zur Operatorenmethode In Der Klassischen Mechanik. Annals of Mathematics

Ornstein's Classification Theorem (1970)

Definition

A Bernoulli shift is a quadruple $(X, \mathcal{B}, \mu, T)$ such that
1. $X = \{1, 2, .., n\}^{\mathbb{Z}}$ for some natural number n
2. $\mathcal{B}$ is the Borel σ-algebra on X
3. μ is a product measure given by a probability distribution $(p_1, ..., p_n)$ with $\sum p_i = 1$
4. T shifts the space: for $x = (x_n)_{n\in\mathbb{Z}}$, $(Tx)_n = x_{n-1}$

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Definition

The Kolmogorov-Sinai entropy of a Bernoulli shift is $-\sum_{i=1}^{n} p_i \log p_i$

Definition

Two Bernoulli shifts $(X, \mathcal{B}, \mu, T)$ and$(Y, \mathcal{C}, \nu, S)$ are isomorphic if there is a measure-preserving map Φ from a μ-measure 1 subset of X onto a ν-measure 1 subset of Y such that $\Phi(Tx) = S\Phi(x)$ for μ-a.e. $x \in X$.

Kolmogorov-Sinai entropy

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If two Bernoulli shifts are isomorphic, then they have the same KS entropy (which is a real number).

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Remark

This theorem was used to disprove a conjecture by von Neumann, asking whether two specific transformations are isomorphic. K-S showed it in the negative by showing their entropies are different.

Kolmogorov-Sinai entropy

Theorem (Kolmogorov-Sinai, 50s)

If two Bernoulli shifts are isomorphic, then they have the same KS entropy (which is a real number).

Theorem (Ornstein 1970)

Two Bernoulli shifts are isomorphic if and only if they have the same entropy.

A classic case of Borel reduction

Point: for each Bernoulli shift, we associate (in a Borel way) a real number, i.e., its entropy, such that the problem of isomorphism is completely reduced to the problem of identity.

Template

Borel map F : Bernoulli shifts $\to \mathbb{R}$, such that

$$X \cong Y \Leftrightarrow F(X) = F(Y)$$

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We now say the Bernoulli shifts are completely *classified* by their entropy.

Results like this in the 70s-90s gave rise to:

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Invariant Descriptive Set Theory

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Definition

Let E_1, E_2 be a definable equivalence relation on standard Borel spaces X_1, X_2, respectively. We say E_1 is *Borel reducible* to E_2 (written as $E_1 \leq_B E_2$) iff there is a Borel function $F : X_1 \to X_2$ such that $uE_1v \Leftrightarrow f(u)E_2f(v)$. Such a function F is called a *Borel reduction* of E_1 to E_2.

Invariant descriptive set theory

Invariant Descriptive Set Theory

Abstract study of the hierarchy of classification problems.

Intuition

A Borel reduction $F : (X_1, E_1) \to (X_2, E_2)$ associates, in a reasonably concrete way, each $x \in X_1$ with a **complete invariant** $y \in X_2$. This way, to know whether $u, v \in X_1$ fall in the same classification, we can just check if they get assigned equivalent invariants. In a number of familiar cases, E_2 is just Identity on some Polish space.

Emergence of a Structural Picture

Theorem (Harrington, Kechris, Louveau, The Glimm-Effros Dichotomy)

If E is a Borel equivalence relation, then one of the following holds:

1. $E \leq_B =_{2^\omega}$
2. $E_0 \leq_B E$

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Origins in operator algebra

- Glimm, J. (1961). *Locally compact transformation groups*, Transactions of the America ematical Society, vol. 101, 124-138
- Effros, E. G. (1965). *Transformation groups and C^*-algebras*, Annals of Mathematics, vol. 81, pp. 38-55

Emergence of a Structural Picture

Theorem (Sierpiński 1917)

There is no Borel reduction of the equivalence relation E_v of being a rational distance away on $\mathbb{R}$ to the identity relation.

First result of Borel equivalence relations theory

...we will have that φ(x)=φ(x') for x−x' rational and φ(x)≠φ(x') for x−x' irrational

Soit maintenant x un nombre réel donné. Designons par $E(x)$ l'ensemble de tous les nombres $x+r$, r étant un nombre rationnel quelconque: on voit sans peine que ce sera un ensemble dénombrable et que nous aurons toujours $E(x)=E(x')$ pour $x-x'$ rationnel et $E(x)\neq E(x')$ pour $x-x'$ irrationnel.

A tout nombre réel donné x correspondra donc un nombre réel $\varphi(x)=f[E(x)]$, et il suit des propriétés de $E(x)$ et $f(E)$ que nous aurons $\varphi(x)=\varphi(x')$ pour $x-x'$ rationnel et $\varphi(x)\neq\varphi(x')$ pour $x-x'$ irrationnel.

Or, je dis que toute fonction $\varphi(x)$ jouissant de cette propriété est non mesurable [1].

I claim that any function having these properties is non-measurable

Sierpiński (1917), Sur quelques problèmes qui impliquent des fonctions non-mesurables.

Proof.

Suppose towards a contradiction that F is a Borel reduction $(x - y \in \mathbb{Q} \Leftrightarrow F(x) = F(y))$. and let b_F be its Borel code
Now force to add a Cohen real c. In $V[c]$, the function F^* coded by b_F still has the same properties as in the assumption of the theorem, by Π^1_1-absoluteness.
But now in $V[c]$, the image $w = F^*(c)$ of the Cohen real under this map remains the same regardless rational translation of c, which implies the value of w is already decided by the weakest condition. So w is already in the ground model, and so its pre-image $F^{-1}(w)$ will contain a real that is a rational distance away from a Cohen real. Contradiction. $\square$

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Emergence of a Structural Picture

Theorem (Sierpiński, 1954)

TFAE:

1. $\aleph_1 \leq |\mathbb{R}|$
2. *There is* $F : \mathbb{R}^\omega \to \mathbb{R}$*, such that* $F(S) \neq$ *any* $S(n)$*, and if* S *and* S' *are permutations of each other, then* $F(S) = F(S')$.

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Sierpiński revealed that the existence of a uniform diagonalizer is a choice principle. It would have been well within the spirit of Sierpiński's program to guess that such a diagonalizer cannot be nice.

Emergence of a Structural Picture

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Theorem (Borel diagonalization theorem. Friedman (1981), *On the necessary use of abstract set theory*)

Define the equivalence relation $\sim$ *on* R^ω*:* $S \sim T$ *iff* $rng(S) = rng(T)$*. Then there is no Borel map* $F : \mathbb{R}^\omega \to \mathbb{R}$ *satisfying*

1. $S \sim T \Rightarrow F(S) = F(T)$
2. $\forall n(F(S) \neq S(n))$

That is, there is no (uniform) Borel diagonalizer.

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This is a canonical way to increase complexity, i.e., the Friedman-Stanley jump of $=_{\mathbb{R}}$. Analogous to the Turing jump.

Birth of IDST: Some Pivotal Publications

- Friedman, H., & Stanley, L. (1989). *A Borel reductibility theory for classes of countable structures.* The Journal of Symbolic Logic, 54(3), 894-914.

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- Kechris, A. S. (1999). *New directions in descriptive set theory.* Bulletin of Symbolic Logic, 5(2), 161-174. (The 1998 Ǵodel Lecture)
- Hjorth's 1998 ICM Lecture, and his two-part Tarski Lectures (2010) (and also numerous other monographs by Hjorth.)

Scaffolding by DST

Set theory provides a scaffold (in Baldwin's sense) a certain parts of mathematics - those that can be codified as definable equivalence relations on Polish spaces (many classification programs, for example).

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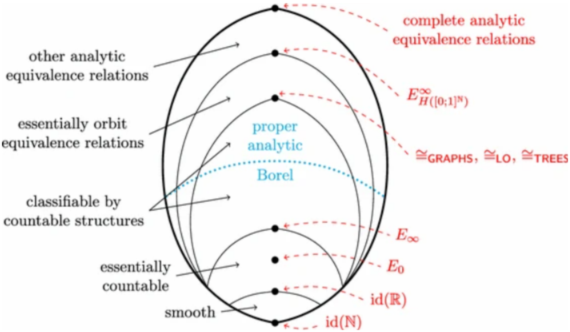

Borel strikes again...

Borel = Tractable

Ros, 2021

"As a matter of fact, the vast majority of classification problems naturally occurring in mathematics falls, up to a suitable coding procedure, inside this class [of analytic equivalence relations]. However, equivalence relations of this kind may be very complicated and **intractable**, so **when possible it is customary to restrict the attention** to the following strictly smaller class [of Borel equivalence relations]."

Generalized Church's Thesis

Foreman and Gorodetski, 2022

A problem can be solved with inherently countable techniques just in case it can be viewed as Borel in a Polish Space.

Theorem (Foreman et al., 2011, *Annals of Mathematics*)

The isomoprhism relation between ergodic measure-preserving transformations is not Borel (relative to natural, suitable background space).

Barrier Exposure: modern articulations

Foreman et al., 2011

"This result explains, perhaps, why the problem of determining whether ergodic transformations are isomorphic or not has proven to be so **intractable** ... [the theorem] can be interpreted as saying that there is no method or protocol that involves a countable amount of information and countable number of steps that reliably distinguishes between nonisomorphic ergodic measure preserving transformations. We view this as a **rigorous way of saying that the classification problem for ergodic measure preserving transformations is intractable.**"

Another example: Smale's Program

Smale, ICM 1962

Classify the diffeomorphisms of a manifold M by topological conjugacy. That is, find reductions of the equivalence relation $f \sim_{top} g$ for $f, g \in \mathrm{Diff}(M)$, where $f \sim_{top} g$ iff there is a homeomorphism $h : M \to M$ such that $h \circ f = g \circ h$.

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Theorem (Foreman, Gorodetski)

This equivalence relation is not Borel (relative to natural, suitable background space)

Foreman and Gorodetski, 2022

"The main result of this paper is that Smale's Program is hopeless in the rigorous sense"

Barrier Exposure

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- Early on, this materialized in the form of revealing the unprovability of certain questions about projective sets.
- But it truly took form by way of the vast amount of anti-classification results, made available in the framework of invariant descriptive set theory.

Summary

Takehome message

DST's venture into definable equivalence has come to form a scaffold for particular areas of mathematics, and this has provided a peculiar kind of guidance (perhaps negatively, yet still productive), which is best characterized as **Barrier Exposure**.

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- This sits in contrast to the kind of Productive Guidance that category theory and model theory provide, which is more about providing a framework for thinking about and organizing mathematics, ultimately assisting in producing new theorems.

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- This sits in contrast to the kind of Productive Guidance that category theory and model theory provide, which is more about providing a framework for thinking about and organizing mathematics, ultimately assisting in producing new theorems.
- This foundational job is best appreciated in the hindsight of the modern development of Borel equivalence relations theory.
- It explains why certain research programs have been stalled or abandoned, by pinpointing the fundamental obstacles with one's problems and/or methods.

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DST's venture into definable equivalence has come to form a scaffold for particular areas of mathematics, and this has provided a peculiar kind of guidance (perhaps negatively, yet still productive), which is best characterized as **Barrier Exposure**.

Finally, curious meta-observation

Unlike the usual stories in the philosophy of set theory, this one doesn't involve any large cardinals, forcing, inner models, or axiom justification at all!

The End

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